PHD

Symmetric R-spaces: A submanifold geometry and transformation theory

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Symmetric R-spaces: Submanifold Geometry and Transformation Theory

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Summary

In this thesis we investigate applications of the theory of symmetric R-spaces: conjugacy classes of height 1 parabolic subalgebras of a semisimple Lie algebra. We study the submanifold theory of symmetric R-spaces firstly in the form of the conformal geometry of line congruences in $\mathbb{P}^3$ viewed as maps into the Klein quadric $\mathbb{P}(\mathcal{L}^{3,3})$, and secondly in terms of the isothermic submanifolds of arbitrary symmetric R-spaces. We also study the application of symmetric R-spaces to the transformation theory of certain geometric objects, where elements of symmetric R-spaces are seen to generate Bäcklund-type transforms in the theory of loop group dressing actions.

Mathematics may be defined as the subject in which we never know
what we are talking about, nor whether what we are saying is true.

Bertrand Russell

Analytical geometry has never existed. There are only people who do linear geometry badly, by taking co-ordinates, and they call this analytical geometry. Out with them!

Jean Dieudonné

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Introduction

In short this thesis consists of three self-contained studies of some of the applications of parabolic geometry in classical and modern Differential Geometry. In particular we study height 1 *parabolic subalgebras* of semisimple Lie algebras and the conjugacy classes of these, namely the *symmetric R-spaces*. Although parabolic subalgebras are not introduced explicitly until Chapter 3 and symmetric R-spaces until Chapter 5, their influence pervades the thesis. Symmetric R-spaces feature in two distinct ways: in chapters 2, 5 and 6 we consider the geometry of certain submanifolds, while chapters 3, 4 and parts of Chapter 5 discuss transformations of various geometric objects constructed out of maps into symmetric R-spaces. Symmetric R-spaces were introduced to mathematics (without the name or any mention of parabolic subalgebras) by Kobayashi–Nagano in their 1965 discussion of *filtered Lie algebras* [42]. This paper contains the definitive list of symmetric R-spaces of simple type,[1] out of which all others can be built. Symmetric R-spaces are indeed symmetric spaces as their name suggests: if $M = G/P$ is a homogeneous G-space where the elements of M are subalgebras of $\mathfrak{g}$, then M is a symmetric $\tilde{G}$-space, where $\tilde{G}$ is the maximal compact subgroup of G. Conversely if a symmetric space $M = \tilde{G}/K$ with $\tilde{G}$ compact admits a Lie group G of diffeomorphisms strictly larger than $\tilde{G}$, then M is a conjugacy class of height 1 parabolic subalgebras of the Lie algebra $\mathfrak{g}$ of G.

A large number of homogeneous spaces of various types play important roles in this thesis, the symmetric R-spaces comprising only a fraction. In particular we have already seen that symmetric R-spaces have two identities: when viewed as conjugacy classes of parabolic subalgebras they are non-reductive homogeneous spaces, e.g. $\mathbb{P}^{n-1} = \mathbb{P}(\mathbb{R}^n) = \mathrm{SL}(n)/P$, $\mathbb{P}(\mathcal{L}^{p,q}) = \mathrm{SO}(p,q)/Q$, whilst viewed as symmetric spaces they are necessarily reductive, e.g. $\mathbb{P}^{n-1} = \mathrm{SO}(n)/\mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(n-1))$, $\mathbb{P}(\mathcal{L}^{p,q}) = (\mathrm{SO}(p) \times \mathrm{SO}(q))/(Q \cap (\mathrm{SO}(p) \times \mathrm{SO}(q)))$. Other examples of homogeneous spaces which play a role are the non-reductive height 2 R-space of isotropic 2-planes in $\mathbb{R}^{p,q}$ viewed as an $\mathrm{O}(p,q)$-space, and the pseudo-Riemannian symmetric spaces of complementary pairs of height 1 parabolic subalgebras. The variety of

[1]Conjugacy classes of parabolic subalgebras of a *simple* Lie algebra.

the homogeneous spaces involved motivates the discussion in Chapter 1. We take an approach to homogeneous spaces which largely avoids the unsatisfactory process of constantly referring everything to a fixed base point. We describe the tangent bundle to reductive and non-reductive homogeneous spaces and the various incarnations of the *soldering form* β which identifies the tangent bundle to a homogeneous space with a quotient subbundle of the trivial Lie algebra bundle. When a homogeneous space G/H is *reductive*, there exists an $\mathrm{Ad}\,H$-invariant splitting of the Lie algebra $\mathfrak{g}$ of G: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. For such spaces we define the *canonical connection* $\mathcal{D}$ on the trivial Lie algebra bundle $G/H \times \mathfrak{g}$ as the restriction of flat d to the factors $\mathfrak{h}, \mathfrak{m}$. In particular we show that the pull-back of $\mathcal{D}$ by an immersion into a symmetric space is flat iff σ is a *curved-flat*, as described by Ferus–Pedit [31].

We also discuss the theory of *parabolic subalgebras* of semisimple Lie algebras. This theory is also widely referenced in various parts of the thesis, but is best introduced in context (cf. page 5). This chapter serves both as an introduction to notation and as a glossary of repeatedly quoted results.

Chapter 2 is a study of *line congruences*: immersions ℓ of a real 2-manifold Σ into the space of lines in $\mathbb{P}^3 = \mathbb{P}(\mathbb{R}^4)$. The study of line congruences in $\mathbb{R}^n$ is a classical subject, dating back to the late 1800s with Ribaucour and Darboux (vol. 2 of his *Leçons*). Line congruences were extensively investigated by Demoulin, Hammond, Jonas, Tzitzeica, Weingarten and Wilczynski, amongst others, in the early 1900s (e.g. [25, 27, 36, 39, 57, 58, 59, 64]). Eisenhart [28] collected together most of the early results while Finikoff [32, 33] and Lane [46] helped revive interest in the middle of the 20th century by propounding a more modern discussion of line congruences in the projective space setting. The study of line congruences is a classic example of Geometers taking inspiration from the study of PDEs: originally a line congruence was defined as a two parameter family of lines tangent to a *conjugate net*; a surface f with co-ordinates x, y satisfying a *Laplace equation*

$$f_{xy} + af_x + bf_y + cf = 0. \tag{1}$$

Laplace investigated transforming the solutions of scalar equations of the above form and observed that the *Laplace transforms* $f^1 := f_y + af$ and $f^{-1} := f_x + bf$ are also solutions to a hyperbolic PDE of the same form, but with *different* coefficients a, b, c. One could then iterate the process in the hope that the Laplace equation satisfied by one of the Laplace transforms f^j was an equation which was better understood. Retrieving the solution to (1) is then a matter of following iteration formulae. Multiple independent solutions to (1) define a conjugate net in $\mathbb{R}^n$. The 2-parameter family of lines joining f with the Laplace transform f^1 form a *line congruence* ℓ, tangent to the *focal surfaces* f, f^1. Laplace transforms are then transformations of surfaces, indeed the most common classical and modern applications of line congruences are as surface transforms: for example the classical Bäcklund transform of pseudospherical surfaces in $\mathbb{R}^3$. Instead of working in $\mathbb{R}^3$, it is more natural [64] to take four independent solutions to (1) viewed as the homogeneous co-ordinates of a surface in $\mathbb{P}^3$. One can reverse the analysis: a generic 2-parameter family of lines ℓ in $\mathbb{P}^3$ possesses two (possibly complex conjugate) focal surfaces f, g and co-ordinates x, y (again possibly complex conjugate) tracing out conjugate nets on f, g to which ℓ is tangent. The reason that $\mathbb{P}^3$ is the natural home for the study of line congruences is two-fold: firstly in higher dimensions a generic congruence has no focal surfaces, and conversely generic surfaces do not possess conjugate nets. Classical studies of line congruences viewed the focal surfaces as fundamental and used them as the basis for most calculations. In particular the (projective) invariants of a line congruence, those of Laplace $\mathcal{H}$ and Weingarten $\mathcal{W}$ are constructed by consideration of the focal surfaces.

Thus far the discussion is entirely classical. Our contribution is to investigate line congruences from the viewpoint of conformal geometry. In so doing we obtain conformal definitions of the Laplace and Weingarten invariants which require no knowledge of the focal surfaces, or of special co-ordinates. The *Klein correspondence* is a bijection between the space of lines in $\mathbb{P}^3$ and the *Klein quadric* $\mathbb{P}(\mathcal{L}^{3,3})$: the space of null lines with respect to a signature (3,3) inner product on $\mathbb{R}^6$ (and a symmetric R-space). Immersions into projective light-cones are known to possess invariants of a similar flavour to the Laplace and Weingarten invariants. The two of interest to us are the *Willmore density* and the curvature of the weightless normal bundle. It is known (e.g. [29]) that congruences for which the Weingarten invariant vanishes (W-congruences) have flat normal bundle and indeed we see that the curvature of the normal bundle (up to a duality between symmetric 2-tensors and 2-forms) is precisely $\mathcal{W}$. The Willmore density C is seen to be related to the Laplace and Weingarten invariants by

$$\mathcal{H} = \frac{1}{2}(C + \mathcal{W}).$$

The analysis is independent of the signature of the inner product on $\mathbb{R}^6$ and can therefore be viewed as a definition of the Laplace invariant of a generic immersion $\ell : \Sigma^2 \to \mathbb{P}(\mathcal{L})$ into *any* light-cone in $\mathbb{R}^6$ when there is not necessarily any concept of focal surface to fall back on. More is true, for we see that, with respect to the Klein correspondence, the Laplace transforms of ℓ are precisely the null directions of the conformal structure in the weightless normal bundle. The concept of Laplace transform therefore has meaning in other projective light-cones in $\mathbb{P}^5$. In particular we learn about Laplace transforms of sphere congruences in Lie sphere geometry, the geometry of the *Lie quadric* $\mathbb{P}(\mathcal{L}^{4,2})$ which parametrises spheres in S^3. The concept of a W-congruence translates exactly to that of a Ribaucour congruence of spheres: both have flat weightless normal bundles and while a line congruence is W iff the asymptotic lines on its focal surfaces coincide, a sphere congruence is Ribaucour iff the curvature lines on its enveloping surfaces coincide.

We return to a classical description of line congruences for the finale. The Laplace invariant $\mathcal{H}^*$ of the *dual congruence* $\ell^* : \Sigma \to \mathbb{P}(\mathbb{R}^4_*)$, the annihilator of ℓ, is seen to satisfy $\mathcal{H}^* = \frac{1}{2}(C - \mathcal{W})$. Moreover Laplace transforms commute with duality in that $(\ell^*)^1 = (\ell^1)^*$. By calculating the structure equations of a congruence with respect to a natural moving frame of $\mathbb{R}^4$ we deduce a couple of corollaries: a proof of the Demoulin–Tzitzeica theorem [25, 58] on the existence of isothermic-conjugate coordinates on focal surfaces, and a discussion of congruences whose image under the Klein correspondence lies in a linear complex. The Demoulin–Tzitzeica theorem motivates a definition of *isothermic line congruence* and we see that the Laplace transforms of an isothermic congruence are also isothermic. The definition provides a link with our second discussion of submanifolds of symmetric R-spaces: the isothermic submanifolds of Chapter 5.

Chapters 3 and 4 are two sides of the same coin: the abstract theory of loop group dressing of maps by simple factors, followed by the application of this theory to Burstall's $\mathfrak{p}$-*flat maps* [11]. For us a loop group is a group $\mathcal{G}$, under pointwise multiplication, of holomorphic maps of subsets $U \subset \mathbb{P}^1$ into $G^{\mathbb{C}}$ where U is $\mathbb{P}^1$ minus a finite set of points and $G^{\mathbb{C}}$ is the complexification of a compact Lie group. The subgroups of positive and negative loops $\mathcal{G}^+$ and $\mathcal{G}^-$ (those loops holomorphic on $\mathbb{C}$ and near ∞ respectively) are seen to satisfy a version of the Birkhoff factorisation theorem [49] which states that the product sets $\mathcal{G}^+\mathcal{G}^-$ and $\mathcal{G}^-\mathcal{G}^+$ are dense open subsets of the identity component of $\mathcal{G}$. Supposing that we restrict $\mathcal{G}^+$ to loops with $g(0) = \mathrm{Id}$ then, by Liouville's theorem, $\mathcal{G}^+ \cap \mathcal{G}^- = \{\mathrm{Id}\}$ and we can build a *local dressing action* of $\mathcal{G}^-$ on $\mathcal{G}^+$: i.e. $g_- \# g_+ := \hat{g}_+$ where $g_- g_+ = \hat{g}_+ \hat{g}_-$.

The theory of dressing by loop groups has found many applications, mostly to the study of PDEs. Uhlenbeck [60] was one of the first to use the technique, observing that harmonic maps into G correspond to *extended solutions* into a suitable $\mathcal{G}^+$, the local dressing action of $\mathcal{G}^-$ on $\mathcal{G}^+$ then corresponds to transformations of harmonic maps (see also Burstall–Guest [12] and Burstall–Pedit [16]). Dorfmeister–Wu [26] used dressing to investigate transformations of constant mean curvature surfaces in $\mathbb{R}^3$. By imposing various conditions[2] on $\mathcal{G}$, Terng–Uhlenbeck [56] describe how solutions to certain families of PDEs correspond to maps into $\mathcal{G}^+$ and how the dressing action generates new solutions. Sadly the action # is in general extremely difficult to calculate (a Riemann–Hilbert problem). However there exist certain elements (Uhlenbeck's elements of *simplest type*) of $\mathcal{G}^-$ whose dressing action is not only easily computable, but computable algebraically, in terms only of g_- and the values of g_+ at at most two points $\alpha, \beta \in \mathbb{C}$. Following Terng–Uhlenbeck [56] we will refer to such elements as the *simple factors*. As they explain, the classical Bäcklund transforms of the Sine–Gordon equation can be described in terms of dressing by simple factors: for this reason transforms by simple factors are often referred to as transforms of *Bäcklund-type*. Our contribution to the study of loop groups is to provide a concrete algebraic definition of simple factor when G is a semisimple Lie group and to calculate their dressing action explicitly. It is here that the symmetric R-spaces make a second appearance, for the simple factors are seen to be parametrised by pairs of *complementary parabolic subalgebras* with Abelian nilradical. One of the simplest examples of a parabolic subalgebra is the infinitesimal stabiliser of a null line ℓ in $\mathbb{R}^{p,q}$ under the action of $\mathfrak{so}(p,q)$. The parabolic subalgebra $\mathrm{stab}(\ell)$ has nilradical (perpendicular space with respect to the Killing form) the subalgebra

$$\mathrm{stab}(\ell)^\perp = \{A \in \mathfrak{so}(p,q) : A\ell = 0,\ A\ell^\perp \subset \ell,\ A\mathbb{R}^{p,q} \subset \ell^\perp\}. \tag{2}$$

With respect to the decomposition $\mathbb{R}^{p,q} = \ell \oplus \ell^\perp/\ell \oplus \mathbb{R}^{p,q}/\ell^\perp$, the nilradical consists of upper triangular matrices and indeed all parabolic subalgebras have this flavour. A complementary parabolic subalgebra to $\mathrm{stab}(\ell)$ is $\mathrm{stab}(\hat{\ell})$ where $\hat{\ell} \neq \ell$. A complementary pair induces a grading of the Lie algebra:

$$\mathfrak{so}(p,q) = \mathrm{stab}(\ell)^\perp \oplus (\mathrm{stab}(\ell) \cap \mathrm{stab}(\hat{\ell})) \oplus \mathrm{stab}(\hat{\ell})^\perp = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}, \tag{3}$$

where $[\mathfrak{g}_j, \mathfrak{g}_k] \subset \mathfrak{g}_{j+k}$, such that $\mathfrak{g}_j := \{0\}$ when $|j| \geq 2$. Given a complementary pair $(\mathfrak{q}, \mathfrak{r})$ of parabolic subalgebras with Abelian nilradical of *any* semisimple Lie algebra and constants $\alpha, \beta \in \mathbb{C}$, the simple factor $p_{\alpha,\beta,\mathfrak{q},\mathfrak{r}}$ is the map

$$p_{\alpha,\beta,\mathfrak{q},\mathfrak{r}} : \mathbb{P}^1 \setminus \{\alpha, \beta\} \to \mathrm{Ad}(\mathfrak{g}^{\mathbb{C}}) : z \mapsto \frac{1-\alpha^{-1}z}{1-\beta^{-1}z}\pi_1 + \pi_0 + \frac{1-\beta^{-1}z}{1-\alpha^{-1}z}\pi_{-1}, \tag{4}$$

where π_j is projection in the Lie algebra with respect to the decomposition (3). If we let $G^{\mathbb{C}}$ be the centre-free group $\mathrm{Ad}(\mathfrak{g}^{\mathbb{C}}) = \mathrm{Int}(\mathfrak{g}^{\mathbb{C}})$, then $p_{\alpha,\beta,\mathfrak{q},\mathfrak{r}}$ is an element of the

[2] 'Reality', $\overline{g(z)} = g(\bar{z})$ with respect to some real form of $G^{\mathbb{C}}$, and/or 'twisting', $\tau g(z) = g(\omega z)$ where τ is a finite order automorphism and ω a root of unity of the same order as τ.

loop group $\mathcal{G}^-$. It turns out that the adjoint representations of all the elements of simplest type as described by Terng and Uhlenbeck are in fact simple factors as in our definition. Having defined and classified the simple factors, we prove that the pointwise dressing action, when defined, of $p_{\alpha,\beta,\mathfrak{q},\mathfrak{r}}$ on $g_+ \in \mathcal{G}^+$ is given by

$$p \# g_+ = p g_+ \hat{p}^{-1},$$

where $\hat{p} = p_{\alpha,\beta,\hat{\mathfrak{q}},\hat{\mathfrak{r}}} \in \mathcal{G}^-$ is also a simple factor. Indeed $\hat{\mathfrak{q}}, \hat{\mathfrak{r}}$, and therefore $p \# g_+$, depend only on $\mathfrak{q}, \mathfrak{r}$ and the values of g_+ at α, β. We conclude the abstract discussion of simple factors with the first of several theorems of *Bianchi permutability*: given two Bäcklund-type transforms by simple factors, there exists a fourth transform that is simultaneously a Bäcklund-type transform of the first two and moreover the fourth transform is computable without resorting to calculus. It may be that there exist further simultaneous Bäcklund-type transforms, but for us a Bianchi quadrilateral always has the simultaneous transform computable 'without quadratures' (e.g. Eisenhart [28] p.143, 211).

Chapter 3 also includes a discussion of loop group dressing for loops twisted by certain higher-order automorphisms τ of G. The expression for the simple factors (4) is valid only when loops are untwisted or twisted by an involution. For higher order τ the analysis fails, since any twisted $g \in \mathcal{G}^-$ with a pole at $\alpha \in \mathbb{C} \setminus \{0\}$ necessarily has poles at $\alpha\omega^j$ for $j = 0, \ldots, \mathrm{Ord}(\tau) - 1$. However Terng–Uhlenbeck [56] provide one example of dressing in the presence of a higher-order twisting: Bäcklund-type transforms of the Kuperschmidt–Wilson hierarchy. We observe that the τ in question is a *Coxeter automorphism* of $\mathfrak{sl}(n, \mathbb{C})$, which has order n. Terng–Uhlenbeck's simple factors in this context have an interesting description in terms of root systems in $\mathfrak{sl}(n)$ which are suitably permuted by τ, the construction of which can be generalised to other simple Lie algebras. We calculate the elements of simplest type for the orthogonal algebras and show that, in contrast to $\mathfrak{sl}(n)$ where simple elements are fractional *linear*, the simplest elements for $\mathfrak{so}(n)$ are *quadratic* fractional. One implication of this is that the 'usual' simple factor dressing proof fails.

Chapter 4 describes the application of the loop group dressing theory of Chapter 3 to Burstall's $\mathfrak{p}$-flat maps [11]. $\mathfrak{p}$-flat maps were originally introduced in order to study Christoffel pairs of isothermic surfaces in S^n, but their scope is somewhat broader. Suppose $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the symmetric decomposition of a real Lie algebra $\mathfrak{g}$ corresponding to a symmetric space G/K. A $\mathfrak{p}$-flat map is an immersion $\psi : \Sigma \to \mathfrak{p}$, whose tangent space at each point spans an Abelian subalgebra of $\mathfrak{p}$. $\mathfrak{p}$-flat maps possess a 1-parameter integrability: $\psi_t := t\psi$ is $\mathfrak{p}$-flat for any scalar t and satisfies the

Maurer-Cartan equations

$$d(d\psi_t) + \frac{1}{2}[d\psi_t \wedge d\psi_t] = 0,$$

from which we can locally integrate $\Phi_t^{-1}d\Phi_t = d\psi_t$ to find a map $\Phi_t : \Sigma \to G^{\mathbb{C}}$ for every $t \in \mathbb{C}$ such that $\Phi_0 \equiv \mathrm{Id}$. The Φ_t may be gathered into a single map Φ which can be viewed as a map $\Sigma \to \mathcal{G}^+$ into the loop group of positive loops twisted with respect to the symmetric involution on $G^{\mathbb{C}}$ and real with respect to G. We call Φ an *extended flat frame* of ψ. The quotient map $\Phi_t K$ is a *curved flat* for every $t \in \mathbb{R}^\times$ and comprises the *spectral deformation* of $\Phi_1 K$. This 1-parameter integrability is worth stressing since it is this that makes the study of $\mathfrak{p}$-flat maps into an interesting geometric theory. Extended flat frames play the same role for $\mathfrak{p}$-flat maps as Uhlenbeck's [60] *extended solutions* play for harmonic maps. Happily the property of being an extended flat frame is preserved by the dressing action of $\mathcal{G}^-$ and so the dressing action descends to a local action on $\mathfrak{p}$-flat maps $g_- \# \psi = \hat{\psi}$. Moreover when g_- is a simple factor we can calculate this action: we see that $g_- \# \psi - \psi$ is a scalar multiple of the *canonical element* of a pair of parabolic subalgebras defined algebraically from g_- and the value of Φ at a single point.

Dressing $\mathfrak{p}$-flat maps by simple factors corresponds to known transforms in several settings. We concentrate on Schief–Konopelchenko's *Bäcklund transform of O-surfaces* [53]: a derivation of the Fundamental transform of Jonas (more properly the Ribaucour transform [28]). A map of dual O-surfaces is a map $\mathbf{R} : \Sigma \to \mathbb{R}^m \otimes \mathbb{R}^n$ with co-ordinates x_i that are orthogonal on left and right contractions of $\mathbf{R}$ with respect to fixed bases of $\mathbb{R}^m, \mathbb{R}^n$. We make no assumption on the signature of the inner products on $\mathbb{R}^m, \mathbb{R}^n$ although Schief–Konopelchenko restrict to $m = 3$ with a definite metric. By judicious choices of the signature on $\mathbb{R}^n$ they show that many classical surfaces may be described in this context: isothermic surfaces, constant mean and Gauss curvature surfaces, Guichard surfaces, Petot surfaces, etc. Their Bäcklund transform involves solving a large matrix-valued PDE to generate a new map of dual O-surfaces and, by contraction, new surfaces in $\mathbb{R}^m, \mathbb{R}^n$. Depending on the choices of signature one sees transformations of the already listed families of surfaces: in particular they show how the classical Bäcklund transform of pseudospherical surfaces fits into this context. We show that maps of dual O-surfaces correspond to a subset of the $\mathfrak{p}$-flat maps into the orthogonal algebra $\mathfrak{so}(p,q)$ where (p,q) is the signature of the combined inner product on $\mathbb{R}^m \oplus \mathbb{R}^n$. Furthermore the Bäcklund-transform corresponds exactly to the dressing action of certain simple factors on the associated $\mathfrak{p}$-flat maps. In particular this gives us a clean method of obtaining graphical representations of the action of simple factors in certain situations.

In Chapter 5 we return to the submanifold geometry of symmetric R-spaces by studying their *isothermic submanifolds*. Isothermic submanifolds generalise the classical subject of isothermic surfaces in $\mathbb{R}^n$ as pioneered by Bianchi and Darboux. By inverse stereo-projecting the isothermic condition in $\mathbb{R}^n$ into the conformal n-sphere $S^n = \mathbb{P}(\mathcal{L}^{n+1,1})$, we see that $\ell : \Sigma \to S^n$ is isothermic precisely when there exists a closed 1-form η taking values in $\ell \wedge \ell^\perp = \operatorname{stab}(\ell)^\perp \subset \mathfrak{so}(n+1,1)$. As already observed (2), the infinitesimal stabiliser of ℓ is a parabolic subalgebra with Abelian nilradical. Motivated by this we define an *isothermic submanifold* to be an immersion $f : \Sigma \to M$ of a real manifold Σ into a (real or complex) symmetric R-space M for which there exists a closed 1-form η on Σ which takes values in the *bundle* (over Σ) of nilradicals $f^\perp$. Similarly by translating the definitions of Christoffel, Darboux and T-transforms from Burstall's [11] discussion of isothermic surfaces in the conformal n-sphere, we obtain these transforms for general isothermic submanifolds, and the classical interactions between them, for example Bianchi's identity

$$\mathcal{T}_t f^c = \mathcal{D}_{-t}\mathcal{T}_t f.$$

The crucial observation driving Darboux and T-transforms is the existence of a pencil of flat G-connections $\mathrm{d}^t := \mathrm{d} + t\eta$ on any trivial Lie algebra bundle. Although all the classical relations between the three transforms are seen to hold, it is with the discussion of Christoffel and Darboux transforms that the abstract theory of isothermic submanifolds deviates from the classical. The *dual* R-space M^* to M is the conjugacy class of all parabolic subalgebras complementary to some element of M. In the motivating case $M = \mathbb{P}(\mathcal{L}^{n+1,1})$ it is easily seen that $M^* = M$ and so M is *self-dual*. When $M^* \neq M$ we say that M is *non-self-dual*: for example $M = \mathbb{P}^n$ ($n \geq 2$) is non-self-dual with $M^* = G_n(\mathbb{R}^{n+1})$. Given an isothermic submanifold of M, its T-transforms are seen to be isothermic submanifolds of M whilst Christoffel and Darboux transforms are submanifolds of the dual space M^*. The difference between the two situations is best exemplified by the Bianchi permutability of Darboux transforms. In a self-dual symmetric R-space, given a pair f_1, f_2 of Darboux transforms of f, there (generically) exists a simultaneous Darboux transform f_{12} of f_1, f_2 which can be constructed entirely by algebraic considerations: i.e. without 'quadratures'. In the non-self-dual case the question of whether Darboux transform permute is currently open, although we have obtained various theorems which may serve as a partial replacement.

In the second half of Chapter 5 we discuss the symmetric space $Z \subset M \times M^*$ of pairs of complementary parabolic subalgebras and observe the generalisation of [15] that Darboux pairs of isothermic submanifolds are exactly the curved flats in Z. The spectral deformation of a Darboux pair $(f, \hat{f})$ is seen to act by T-transforms on the individual isothermic submanifolds. We abstract slightly the study of curved flats in order to describe a general method of transforming curved flats in any symmetric G-space for which the Lie algebra $\mathfrak{g}^{\mathbb{C}}$ possesses parabolic subalgebras with Abelian nilradical. Since this is seen to correspond exactly to the discussion of dressing $\mathfrak{p}$-flat maps by simple factors as described in Chapter 4, we refer to these transforms as *dressing curved flats*. Applying the dressing theory to the curved flats formed by Darboux pairs of isothermic submanifolds allows us to obtain a second proof of the Bianchi permutability of Darboux transforms in a self-dual M and moreover a Bianchi cube theorem: morally a triple permutability theorem. The correspondence with the dressing of $\mathfrak{p}$-flat maps allows us to recover Burstall's observation that the curved flats frame Christoffel pairs of isothermic submanifolds.

Three examples of symmetric R-spaces and their isothermic submanifolds are considered in Chapter 6. Firstly we investigate the complexified light-cone in $\mathbb{C}^6$ and are able to prove an extension of the Demoulin–Tzitzeica theorem: the Laplace transforms of an isothermic line congruence are isothermic. Not only does this hold for maps into the complex quadric, but also for maps into the three real forms: $S^4 = \mathbb{P}(\mathcal{L}^{5,1})$, the Lie quadric $\mathbb{P}(\mathcal{L}^{4,2})$ and the Klein quadric $\mathbb{P}(\mathcal{L}^{3,3})$. In S^4 this is not particularly interesting since Laplace transforms are necessarily complex. The Lie quadric result is more interesting: since the quadric parametrises 2-spheres in S^3 in the same way that the Klein quadric does for lines in $\mathbb{P}^3$, we see that an isothermic sphere congruence in S^3 with two real enveloping surfaces has real isothermic Laplace transforms. The Klein quadric result applied to line congruences with real focal surfaces recovers the correspondence between the (classical) definition of isothermic line congruence as given in Chapter 2 and the modern definition in Chapter 5. We are also able to see that isothermic submanifolds of higher-dimensional light-cones have, as expected, a flat normal bundle.

The Grassmannians $G_k(\mathbb{R}^n)$ $(k \neq n/2)$ comprise our first detailed investigation of a non-self-dual symmetric R-space. In order to make contact with previous discussions we firstly work in $G_2(\mathbb{R}^n)$: maps $\ell : \Sigma^2 \to G_2(\mathbb{R}^n)$ are line congruences in $\mathbb{P}^{n-1}$. A 2-parameter family of lines ℓ in $\mathbb{P}^{n-1}$, $n > 4$ does not, in general, have any special properties such as focal surfaces, special co-ordinates, Laplace transforms, etc. However, when ℓ is isothermic a remarkable amount of the theory of Chapter 2 is preserved. For generic congruences the existence of a Darboux transform is enough for us to be able to construct special co-ordinates, focal surfaces, Laplace invariants, isothermic Laplace transforms, dual surfaces, etc. and a canonical expression for the closed 1-form η. We see that isothermic line congruences are W-congruences in that there exists a suitable choice of second fundamental form on each focal surface such that the asymptotic directions coincide. More interestingly we observe a duality: given an isothermic congruence of k-planes π which is suitably *maximal*,[3] there is a unique $(n-k)$-plane $\hat{\pi}$ which is isothermic with respect to the *same* closed 1-form η. We therefore have a duality between maximal isothermic submanifolds of $G_k(\mathbb{R}^n)$ and those of the dual R-space $G_{n-k}(\mathbb{R}^n)$. Using this duality we are able to prove a pleasing substitute for the Bianchi permutability of Darboux transforms.

Intriguingly a similar structure is observed in our second example of a non-self-dual symmetric space. As a symmetric R-space, $\mathrm{SO}(n)$ comprises the stabilisers of isotropic n-planes in $\mathbb{R}^{n,n}$, and is self-dual iff n is even. When n is odd we find a duality between maximal isothermic submanifolds of $\mathrm{SO}(n)$ and those of $\mathrm{SO}(n)^*$. The construction of the duality is very similar to that for the Grassmannians, relying in both cases on the fact that the closed 1-form η is aware of greater structure than just the original isothermic submanifold. The similarity of the constructs suggests the existence of a general duality theorem, from which a general replacement for the theorem of Bianchi permutability of Darboux transforms in self-dual symmetric R-spaces would follow. However, at the level of Lie algebras, the construction of the duality is not clear and so a statement of a general theorem for non-self-dual symmetric R-spaces currently eludes us.

[3]There exists a Darboux transform $\hat{\pi}$ such that the image of the derivative of the curved flat $(\pi, \hat{\pi})$ is a maximal semisimple Abelian subalgebra: this is a standard assumption made repeatedly in papers such as [9, 11, 56].

Chapter 1

Homogeneous Geometry and Parabolic Subalgebras

This thesis repeatedly makes use of homogeneous spaces, both reductive and non-reductive, and especially of the identification of the tangent bundle to a homogeneous space with (quotient) subbundles of the trivial Lie algebra bundle. We give a discussion of homogeneous spaces in a manner which allows us to work with homogeneous spaces independently of a choice of base point. Throughout G is a Lie group with Lie algebra $\mathfrak{g}$ and M is a transitive G-space: a manifold on which G acts smoothly and transitively. If $x \in M$, then $\mathrm{Stab}(x) \subset G$ is the stability Lie subgroup of x while $\mathrm{stab}(x) \subset \mathfrak{g}$, the Lie algebra of $\mathrm{Stab}(x)$, is the infinitesimal stabiliser of the tangent space at x under the action of $\mathfrak{g}$. We write trivial bundles over M by underlining the fibre: e.g. $\underline{\mathfrak{g}} = M \times \mathfrak{g}$. The main references for this discussion are [18, 41, 44, 47, 48].

1.1 The Tangent Bundle and the Soldering Form

Let H be the bundle of Lie subgroups of G over M with fibre $H_x = \mathrm{Stab}(x)$. The Lie algebras $\mathfrak{h}_x = \mathrm{stab}(x)$ of H_x are similarly the fibres of a bundle $\mathfrak{h}$ over M. Fix a base point $x \in M$, then $M \cong G/H_x : g \cdot x \mapsto gH_x$ is a *homogeneous space* for *any* x. We will often abuse notation and write G/H to stress the independence of the base point. Similarly statements such as $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$ and operators $P_{\mathfrak{h}} = \mathrm{Proj}_{\mathfrak{h}}$ are to be understood fibrewise: for example if $v \in \Gamma\underline{\mathfrak{g}}$ (a section), then $(\mathrm{Proj}_{\mathfrak{h}} v)_x = \mathrm{Proj}_{\mathfrak{h}_x} v_x$ and so $\mathrm{Proj}_{\mathfrak{h}} v \in \Gamma\mathfrak{h}$. We may also form the quotient bundle $\underline{\mathfrak{g}}/\mathfrak{h}$, whose fibre at x is $\mathfrak{g}/\mathfrak{h}_x$.

Definition 1.1
Let P, N be manifolds and L a Lie group. A principal L-bundle is a surjective submersion $\pi : P \to N$ such that L has a free right action $P \times L \to P$ with orbits the fibres of

π:

$$p \cdot L = \pi^{-1}\{\pi(p)\}.$$

Given a principal L-bundle $\pi : P \longrightarrow N$ and a left L-space F, the associated bundle to P with typical fibre F is the right L-space

$$X = (P \times F)/L = P \times_L F$$

where L acts freely by

$$(p, f) \cdot l = (p \cdot l, l^{-1} \cdot f).$$

The orbit of L through (p, f) is written $[p, f] = [pl, l^{-1}f]$.

Let $P = G$ and $N = M$ in the above definition. The surjectivity of π means that $L \longrightarrow G \xrightarrow{\pi} M$ is exact and so $M \cong G/L$ is a homogeneous space. Taking $x_0 \in M$ to be the identity coset gives $L = H_{x_0}$. It is clear that the bundle $\mathfrak{h}$ is equivariant in the sense that $\operatorname{Ad} g\mathfrak{h}_x = \mathfrak{h}_{g\cdot x}$ and so

$$[g, \xi] \mapsto (g \cdot x, \operatorname{Ad} g\xi) \tag{1.1}$$

is an isomorphism $\mathfrak{h} \cong G \times_{H_x} \mathfrak{h}_x$ for *any* $x \in M$.

Lemma 1.2
Let $M \cong G/H_x$ be homogeneous. The map $\pi : G \longrightarrow M : g \mapsto g \cdot x$ is a principal H_x-bundle.

Proof H_x acts on G by right multiplication such that

$$(g \cdot H_x) \cdot x = g \cdot (H_x \cdot x) = g \cdot x = \pi(g).$$

Thus $g \cdot H_x = \pi^{-1}\{\pi(g)\}$. It remains to show that π is a submersion. Consider first $g = \mathrm{Id}$. $\mathfrak{g}$ is isomorphic to T_1G via $\xi \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \exp t\xi$. Thus

$$\xi \mapsto \mathrm{d}\pi_1 \left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \exp t\xi \right) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \pi(\exp t\xi) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \exp(t\xi) \cdot x$$

is onto T_xM with kernel $\mathfrak{h}_x$. $\mathrm{d}\pi_1$ is surjective and so, by left translation, is $\mathrm{d}\pi_g$ for all g.

■

Since the above holds for any $x \in M$, we have an isomorphism $TM \cong G \times_{H_x} \mathfrak{g}/\mathfrak{h}_x$ which, by the equivariance (1.1) of $\mathfrak{h}$, reads

$$TM \cong \underline{\mathfrak{g}}/\mathfrak{h}.$$

This identification is given by a family of isomorphisms

$$\beta : T_xM \to \mathfrak{g}/\mathfrak{h}_x : X_x = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(t\xi) \cdot x \mapsto \xi + \mathfrak{h}_x \tag{1.2}$$

which we will refer to as the *soldering form*.[1] β will be viewed as a bundle-valued 1-form: $\beta \in \Omega^1_M \otimes \underline{\mathfrak{g}}/\mathfrak{h}$. It is clear from (1.2) that β is equivariant:

$$g^*\beta = \mathrm{Ad}\, g\beta. \tag{1.3}$$

Furthermore (1.2) tells us that

$$X_x = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(t\beta(X_x)) \cdot x. \tag{1.4}$$

It follows that for any section $v \in \Gamma\underline{\mathfrak{g}}$ and vector field $X \in \Gamma TM$ we have[2]

$$\mathrm{d}v(X) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(t\beta(X))v = \mathrm{ad}(\beta(X))v \pmod{[\mathfrak{h}, v]}. \tag{1.5}$$

Our bundle approach to the stability groups H_x has already born fruit, for we can now almost entirely avoid talking about identifications (1.1) of associated bundles with trivial bundles.

For certain homogeneous spaces, the soldering form takes on extra structure.

Definition 1.3
A homogeneous space G/H_x is reductive *if $\mathfrak{g} = \mathfrak{h}_x \oplus \mathfrak{m}_x$ for some* Ad H_x*-invariant $\mathfrak{m}_x$ known as a* reductive factor *of G/H_x.*

Since each $\mathfrak{m}_x$ is $\mathrm{Ad}\, H_x$ invariant we may left translate a reductive factor $\mathfrak{m}_x$ to form a bundle $\mathfrak{m}$ over M for which each fibre $\mathfrak{m}_x$ is a reductive factor for G/H_x as in the definition. Thus $\underline{\mathfrak{g}} = \mathfrak{h} \oplus \mathfrak{m}$ is a bundle decomposition. Since $\mathfrak{g}/\mathfrak{h}_x \cong \mathfrak{m}_x$ as H_x-spaces we say that $\underline{\mathfrak{g}}/\mathfrak{h}$ and $\mathfrak{m}$ are bundle isomorphic as H-spaces. Under this identification the soldering form β is now a genuine $\mathfrak{g}$-valued 1-form on M, rather

[1] Kobayashi [41] calls β a *form of sodure* (=welding). *Soldering form* is a translation used by Physicists.
[2] Another fibrewise statement: if $X \in T_xM$ then we quotient out by $[\mathfrak{h}_x, v]$.

than being quotient valued. Indeed for each $X \in T_xM$ there is a unique $\xi \in \mathfrak{m}_x$ satisfying

$$X = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(t\xi) \cdot x, \qquad \beta(X) = \xi, \tag{1.6}$$

and so $TM \cong \underline{\mathfrak{m}}$ ($\cong G \times_{H_x} \mathfrak{m}_x, \forall x \in M$), whence TM is identified with a genuine subbundle of $\underline{\mathfrak{g}}$. In the presence of a reductive factor the identification (1.5) now reads

$$\mathrm{d}v(X) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(t\beta(X))v = \mathrm{ad}(\beta(X))v. \tag{1.7}$$

When M is reductive, the soldering form β is referred to as the *Maurer–Cartan form* of M, for when $M = G$, (1.4) says that β is the right Maurer–Cartan form of G. We will tend to write $\mathcal{N}$ for β when M is reductive.

1.2 The Canonical Connection

Reductive homogeneous spaces M posses extra structure in the form of an invariant connection on $TM \cong \underline{\mathfrak{m}}$. Our interest is not so much with TM as with the entire trivial bundle $\underline{\mathfrak{g}}$, so for us this connection will also be defined on $\underline{\mathfrak{h}}$. Since $\mathcal{N} \in \Omega^1_M \otimes \mathfrak{g}$, we may define a connection $\mathcal{D} = \mathrm{d} - \mathrm{ad}\,\mathcal{N}$ on $\underline{\mathfrak{g}}$. This is the covariant derivative on $\underline{\mathfrak{g}}$ induced by the *canonical connection* of M as the following proposition shows.[3]

Proposition 1.4
Fix any base point $x \in M$. The connection form α of $\mathcal{D} = \mathrm{d} - \mathcal{N}\cdot$ under the isomorphism $\underline{\mathfrak{g}} \cong G \times_{H_x} \mathfrak{g}$ is $\mathrm{Proj}_{\mathfrak{h}_x}\theta$, where θ is the left Maurer–Cartan form of G.

The proposition is the standard definition of the canonical connection on any associated bundle $G \times_{H_x} V \cong M \times V$ for which V is a representation of G. The following proof is valid, with a little relabelling, in any such context.

Proof Under the identification $G \times_{H_x} \mathfrak{g} \cong M \times \mathfrak{g}$, a section f of $\underline{\mathfrak{g}}$ corresponds to a map $\hat{f} : G \to \mathfrak{g}$ via

$$\hat{f}(g) = \mathrm{Ad}\, g^{-1} f(\pi(g)). \tag{1.8}$$

Under the same identification, $\mathcal{D}f \in \Omega^1_M \otimes \mathfrak{g}$ becomes a $\mathfrak{g}$-valued 1-form on G:

$$\mathrm{d}\hat{f} + \alpha \cdot \hat{f} = \mathrm{Ad}\, g^{-1}(\pi^*\mathcal{D})f.$$

[3] This is similar to Proposition 1.1 of [18] in reverse.

However by differentiating (1.8) we see that

$$\begin{aligned}
\mathrm{d}\hat{f} &= -g^{-1}\mathrm{d}gg^{-1}\cdot f\circ\pi + \operatorname{Ad}g^{-1}\mathrm{d}f\circ\mathrm{d}\pi \\
&= -\theta\cdot\hat{f} + \operatorname{Ad}g^{-1}\pi^*\mathrm{d}f,
\end{aligned}$$

and so

$$\begin{aligned}
\alpha\cdot\hat{f} &= \operatorname{Ad}g^{-1}\pi^*(\mathrm{d}f - \mathcal{N}\cdot f) - \mathrm{d}\hat{f} \\
&= \theta\cdot\hat{f} - \operatorname{Ad}g^{-1}\circ(\mathcal{N}\circ\mathrm{d}\pi)\cdot(\operatorname{Ad}g\hat{f}) \\
&= (\theta - \operatorname{Ad}g^{-1}(\pi^*\mathcal{N}))\cdot\hat{f}.
\end{aligned}$$

However if $X_g \in T_gG$, then

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\exp(t\mathcal{N}(\mathrm{d}\pi X_g))\pi(g) = \mathrm{d}\pi(X_g) &= \mathrm{d}\pi\circ\mathrm{d}L_g\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\exp(t\theta_X) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} g\exp(t\theta_X)g^{-1}\pi(g) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\exp(t\operatorname{Ad}g\theta_X)\pi(g)
\end{aligned}$$

and so $(\pi^*\mathcal{N})_g = \operatorname{Ad}g\operatorname{Proj}_{\mathfrak{m}_x}\theta$. Thus $\alpha = \operatorname{Proj}_{\mathfrak{h}_x}\theta$ as required.

■

Remarks: The proof is a little overdramatic, for once one observes the expression for the pull-back of $\mathcal{N}$ by π and that the connection form of d is just θ then the theorem simply says

$$\mathrm{d} = \mathcal{D} + \mathcal{N}\cdot \iff \theta = \operatorname{Proj}_{\mathfrak{h}_x}\theta + \operatorname{Proj}_{\mathfrak{m}_x}\theta.$$

It is clear that the canonical connection is metric for any invariant metric B on $\mathfrak{g}$, since each $\mathcal{N}_X$ acts by inner derivations. In particular $\mathcal{D}$ is metric for the Killing form of $\mathfrak{g}$.

Definition 1.5

Let G be a Lie group endowed with an involution τ and $H_0 \subset G$ a closed Lie subgroup, open in the fixed set of τ. The homogeneous space G/H_0 is termed symmetric.[4]

The involution τ imparts extra structure on the Lie algebra $\mathfrak{g}$ of G. The derivative $\mathrm{d}\tau_1$ (from now on also called τ)[5] splits $\mathfrak{g}$ into $\pm$-eigenspaces

$$\mathfrak{g} = \mathfrak{h}_0 \oplus \mathfrak{m}_0, \tag{1.9}$$

[4] For a more thorough discussion of symmetric spaces see e.g. [37].

[5] Since if τ is conjugation by ρ in a matrix group G, then $\mathrm{d}\tau_1 = \operatorname{Ad}\rho$ is essentially the same map.

where $\mathfrak{h}_0$ is the Lie algebra of H_0. Since τ is an automorphism it is clear that $\operatorname{ad}(\tau g) = \tau \circ \operatorname{ad} g \circ \tau^{-1}$, from which we have

$$[\mathfrak{h}_0, \mathfrak{h}_0], [\mathfrak{m}_0, \mathfrak{m}_0] \subset \mathfrak{h}_0, \qquad [\mathfrak{h}_0, \mathfrak{m}_0] \subset \mathfrak{m}_0, \tag{1.10}$$

and so $\mathfrak{m}_0$ is a reductive factor for G/H_0. A splitting (1.9) of a Lie algebra $\mathfrak{g}$ satisfying (1.10) is termed a *symmetric decomposition*. Observe that $\operatorname{ad} \mathfrak{h}_0 \circ \operatorname{ad} \mathfrak{m}_0$ permutes $\mathfrak{h}_0, \mathfrak{m}_0$ and is therefore trace-free. Hence $\mathfrak{m}_0 = \mathfrak{h}_0^\perp$ with respect to the Killing form. Also note that a symmetric decomposition recovers τ by specifying its $\pm$-eigenspaces $\mathfrak{h}_0, \mathfrak{m}_0$.

The subscript 0 reflects the fact that we have implicitly chosen a base point x_0 (the identity coset of H_0). One may remove this choice by letting τ be the bundle of involutions with fibre $\tau_{g \cdot x_0} = \operatorname{Ad} g \circ \tau_{x_0} \circ \operatorname{Ad} g^{-1}$. The eigenspaces of τ are therefore the bundles $\mathfrak{h}, \mathfrak{m} = \mathfrak{h}^\perp$ where $\mathfrak{h}_x = \operatorname{stab}(x)$. We call

$$\underline{\mathfrak{g}} = \mathfrak{h} \oplus \mathfrak{m} \tag{1.11}$$

a symmetric decomposition of the trivial Lie algebra bundle and G/H a symmetric space. We will refer to this as the *bundle* definition of a symmetric space. Whenever we need to work with τ, or an explicit symmetric decomposition, it will be explained which definition we are working with. For the remainder of this section we stick to the bundle definition.

Proposition 1.6
Let $M = G/H$ be reductive and let $D = P_\mathfrak{h} \circ \mathrm{d} \circ P_\mathfrak{h} + P_\mathfrak{m} \circ \mathrm{d} \circ P_\mathfrak{m}$ be the connection on $\underline{\mathfrak{g}}$ given by restriction to $\mathfrak{h}, \mathfrak{m}$. Then D is the canonical connection iff M is a symmetric space.

Proof Consider the projection maps $P_\mathfrak{h}, P_\mathfrak{m}$. If $X \in T_xM$ and $v \in \Gamma\underline{\mathfrak{g}}$, (1.7) tells us that

$$\begin{aligned} \mathrm{d}P_\mathfrak{h}(X)v &= \mathrm{d}(P_\mathfrak{h}v)(X) - P_\mathfrak{h}\mathrm{d}v(X) = [\mathcal{N}_X, P_\mathfrak{h}v] - P_\mathfrak{h}[\mathcal{N}_X, v] \\ &= [\operatorname{ad}\mathcal{N}_X, P_\mathfrak{h}]v, \end{aligned}$$

and similarly $\mathrm{d}P_\mathfrak{m}(X) = [\operatorname{ad}\mathcal{N}_X, P_\mathfrak{m}]$. We now have

$$\begin{aligned} D &= \mathrm{d} - [\operatorname{ad}\mathcal{N}, P_\mathfrak{h}] \circ P_\mathfrak{h} - [\operatorname{ad}\mathcal{N}, P_\mathfrak{m}] \circ P_\mathfrak{m}. \\ &= \mathrm{d} - \operatorname{ad}\mathcal{N} + P_\mathfrak{h} \circ \operatorname{ad}\mathcal{N} \circ P_\mathfrak{h} + P_\mathfrak{m} \circ \operatorname{ad}\mathcal{N} \circ P_\mathfrak{m}. \end{aligned}$$

Thus $D = \mathcal{D}$ iff

$$P_\mathfrak{h} \circ \operatorname{ad}\mathcal{N} \circ P_\mathfrak{h} = 0 = P_\mathfrak{m} \circ \operatorname{ad}\mathcal{N} \circ P_\mathfrak{m}.$$

Since ad $\mathcal{N}$ is onto $\mathfrak{m}$, the first of these is equivalent to $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ which, since G/H is reductive, we already know. The second holds iff $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$: that is $M = G/H$ is symmetric.

∎

In this thesis we will never consider homogeneous spaces M intrinsically, rather we will only concern ourselves with immersions $\sigma : \Sigma \to M$ of some (usually simply connected) manifold Σ. The soldering form β identifies the image of the derivative of σ with a (quotient) subbundle of $M \times \mathfrak{g}$. We may therefore pull-back all bundles to Σ and change notation accordingly:

$$\underline{\mathfrak{g}} := \sigma^*(M \times \mathfrak{g}) = \Sigma \times \mathfrak{g},$$

Underlining now refers to trivial bundles over Σ rather than M as previously. Similarly we denote the pull-backs of the soldering form and the canonical connection (for reductive M) by β, ($\mathcal{N}$,) $\mathcal{D}$, etc. That is

$$\mathrm{d}\sigma(= \sigma^*\beta^{\mathrm{old}}) = \beta : T\Sigma \to \underline{\mathfrak{g}}/\mathfrak{h} \subset \underline{\mathfrak{g}} = \Sigma \times \mathfrak{g}, \tag{1.12}$$

while $\mathcal{D}$ is a connection on $\Sigma \times \mathfrak{g}$.

We conclude with a discussion of curved flats in a symmetric space in terms of the canonical connection.

Definition 1.7
A curved flat [31] is an immersion $\sigma : \Sigma \to M$ into a symmetric space for which the image of each tangent space $\mathrm{d}\sigma(T_s\Sigma) = \mathcal{N}(T_s\Sigma) \subset \mathfrak{g}$ is an Abelian subalgebra.

Proposition 1.8
Let σ be a map into a symmetric space of semisimple type (G/H where G is semisimple), then σ is a curved flat iff the canonical connection $\mathcal{D}$ along σ is flat.

Proof The Abelian condition is equivalent to $[\mathcal{N} \wedge \mathcal{N}] = 0$.[6] However, the flatness of $\mathrm{d} = \mathcal{D} + \mathcal{N}\cdot$ reads[7]

$$R^{\mathrm{d}} = 0 = R^{\mathcal{D}} + \mathrm{d}^{\mathcal{D}}\mathcal{N} + \frac{1}{2}[\mathcal{N} \wedge \mathcal{N}], \tag{1.13}$$

which, by collecting terms, gives

$$R^{\mathcal{D}} + \frac{1}{2}[\mathcal{N} \wedge \mathcal{N}] = 0 = \mathrm{d}^{\mathcal{D}}\mathcal{N}.$$

[6]Use the Lie bracket to evaluate products: $[\mathcal{N} \wedge \mathcal{N}](X, Y) = [\mathcal{N}_X, \mathcal{N}_Y] - [\mathcal{N}_Y, \mathcal{N}_X] = 2[\mathcal{N}_X, \mathcal{N}_Y]$.
[7]Recall that $\mathrm{d}^{\mathcal{D}}\mathcal{N}_{X,Y} = \mathrm{d}^{\mathcal{D}}_X\mathcal{N}_Y - \mathrm{d}^{\mathcal{D}}_Y\mathcal{N}_X - \mathcal{N}_{[X,Y]} = \mathrm{d}^{\mathcal{D}}_X \circ \mathcal{N}_Y - \mathcal{N}_Y \circ \mathrm{d}^{\mathcal{D}}_X - \cdots$.

Thus $R^{\mathcal{D}} = 0 \iff \mathrm{ad}[\mathcal{N} \wedge \mathcal{N}] = 0 \iff [\mathcal{N} \wedge \mathcal{N}] = 0$ since ad is an isomorphism. It follows that σ is a curved flat iff $\mathcal{D}$ is flat.

■

Caution is required when applying the above result. In the more standard discussion of the canonical connection, $\mathcal{D}$ is a connection only on $\mathfrak{m} \cong TM$ rather than on the whole of $\underline{\mathfrak{g}}$. In such a case one has the restricted condition

$$R^{\mathcal{D}} = 0 \iff [[\mathcal{N} \wedge \mathcal{N}], \mathfrak{m}] = 0.$$

Assuming that each $\mathcal{N}(T_s\Sigma)$ is conjugate to a fixed subalgebra $\mathfrak{a}$ makes the above condition algebraic and, in most contexts—in particular for G semisimple (Lemma 1 in [31])—implies $[\mathfrak{a}, \mathfrak{a}] = 0$. This explains the caveat given in the discussion of curved flats in [11]: "Under mild conditions ... the curvature operator of the canonical connection on G/H vanishes on each $\bigwedge^2 \mathrm{d}\varphi(T_s\Sigma)$."

1.3 Parabolic Subalgebras

Repeated use is made of this section in various parts of the thesis. The discussion of simple factors in chapters 3 and 4 is mainly concerned with parabolic subalgebras of complex Lie algebras, albeit occasionally with some reality condition, while chapters 5 and 6 require a thorough discussion of both real and complex parabolic subalgebras. Being not widely discussed in the literature, it is useful to gather together all the results that will be of use in later parts of this thesis. We will only require results for parabolic subalgebras with Abelian nilradical, although all statements in this section are easily generalisable to arbitrary height. When it is straightforward to do so we give general proofs. For further discussion of parabolic subalgebras see e.g. [18, 21, 40, 55]. I am grateful to Fran Burstall for his notes on this subject and from which many of the arguments in this section originated.

Definition 1.9 (Grothendieck)
Let $\mathfrak{g}$ be a (real or complex) semisimple Lie algebra. A parabolic subalgebra *of $\mathfrak{g}$ is a subalgebra $\mathfrak{q}$ such that $\mathfrak{q}^{\perp}$ is a nilpotent subalgebra of $\mathfrak{q}$. We will refer to $\mathfrak{q}^{\perp}$ as the* nilradical *of $\mathfrak{q}$.*

The more standard definition of a parabolic subalgebra is that its complexification $\mathfrak{q}^{\mathbb{C}}$ contains a Borel (maximal solvable) subalgebra of $\mathfrak{g}^{\mathbb{C}}$. The definition above is far more useful to our needs: the two were shown to be equivalent by Grothendieck [35].

Let $\mathfrak{q} \subset \mathfrak{g}$ be parabolic. The nilpotency of $\mathfrak{q}^{\perp}$ means that its central descending series terminates. If we set

$$\mathfrak{q}^{(0)} := \mathfrak{q}, \quad \mathfrak{q}^{(1)} := \mathfrak{q}^{\perp}, \quad \mathfrak{q}^{(j)} := \begin{cases} [\mathfrak{q}^{(1)}, \mathfrak{q}^{(j-1)}], & j > 1, \\ (\mathfrak{q}^{(-j+1)})^{\perp}, & j < 0, \end{cases} \tag{1.14}$$

then $\exists n$ (the *height* of $\mathfrak{q}$) such that $\mathfrak{q}^{(n)} \neq \{0\}$, $\mathfrak{q}^{(n+1)} = \{0\}$. Then $\mathfrak{g} = \mathfrak{q}^{(-n)}$ and $\mathfrak{q}^{(j-1)} \supsetneq \mathfrak{q}^{(j)}$, $\forall j = -n+1, \ldots, n$. For $p_i \in \mathfrak{q}$, $p^{\perp} \in \mathfrak{q}^{\perp}$, we have $B([p_1, p^{\perp}], p_2) = -B(p^{\perp}, [p_1, p_2]) = 0$, where B is the (non-degenerate) Killing form, and so $[\mathfrak{q}, \mathfrak{q}^{\perp}] \subset \mathfrak{q}^{\perp}$. A straightforward induction argument shows that

$$[\mathfrak{q}^{(j)}, \mathfrak{q}^{(k)}] \subset \mathfrak{q}^{(j+k)} \tag{1.15}$$

where, by convention, we set $\mathfrak{q}^{(j)} = \{0\}$ for $|j| > n$. $\mathfrak{g}$ is therefore a filtered algebra.

A single parabolic subalgebra does not induce much structure on $\mathfrak{g}$: a pair however will generally induce a grading on $\mathfrak{g}$ from the two filterings (1.15).

Definition 1.10
A pair of parabolic subalgebras $(\mathfrak{q}, \mathfrak{r})$ *are* complementary *if* $\mathfrak{g} = \mathfrak{q}^{(-j)} \oplus \mathfrak{r}^{(j+1)}$, $\forall j \geq 0$.

By (1.14) it is immediate that $\dim \mathfrak{r}^{(j)} = \dim \mathfrak{q}^{(j)}$ and so complementary parabolic subalgebras have the same height. Complementarity is clearly a symmetric relation by taking Killing perps in the definition. It is not however an equivalence relation since $\mathfrak{q}$ is not complementary to itself. Real non-compact semi-simple Lie algebras always have complementary pairs of parabolic subalgebras since such algebras admit Cartan decompositions: symmetric decompositions (Definition 1.5) $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where the Killing form of $\mathfrak{g}$ is negative-definite on $\mathfrak{k}$ and positive-definite on $\mathfrak{p} = \mathfrak{k}^{\perp}$; indeed if τ is the corresponding *Cartan involution* (its $\pm$-eigenspaces are $\mathfrak{k}, \mathfrak{p}$) then $(\mathfrak{q}, \tau\mathfrak{q})$ are complementary for any parabolic subalgebra $\mathfrak{q}$. It should be observed that real compact algebras have no non-trivial[8] parabolic subalgebras, since the nilradical $\mathfrak{q}^{\perp}$ would necessarily be isotropic for the (negative-definite) Killing form.

Proposition 1.11
Complementary parabolic subalgebras $\mathfrak{q}, \mathfrak{r}$ *make* $\mathfrak{g}$ *into a graded algebra*

$$\mathfrak{g} = \bigoplus_{|j| \leq n} \mathfrak{g}_j = \bigoplus_{|j| \leq n} \mathfrak{q}^{(j)} \cap \mathfrak{r}^{(-j)}$$

where n *is the height of* $\mathfrak{q}, \mathfrak{r}$.

[8] Any semisimple Lie algebra is trivially parabolic with nilradical $\{0\}$.

Proof Let $j \geq 0$, define $\mathfrak{c}_j = \mathfrak{q}^{(-j)} \cap \mathfrak{r}^{(-j)}$ and notice that

$$\mathfrak{g}_0 := \mathfrak{q} \cap \mathfrak{r} = \mathfrak{c}_0 \subset \mathfrak{c}_1 \subset \cdots \subset \mathfrak{c}_n = \mathfrak{g}.$$

It is easy to see that $\mathfrak{c}_j \cap \mathfrak{c}_j^{\perp} = \{0\}$, so that B is non-degenerate on each $\mathfrak{c}_j$. We therefore have

$$\begin{aligned}
\mathfrak{c}_j &= \mathfrak{c}_{j+1} \oplus (\mathfrak{c}_{j+1}^{\perp} \cap \mathfrak{c}_j), \ \forall j \in (1, n), \\
\Rightarrow \mathfrak{g} &= \mathfrak{c}_0 \oplus \bigoplus_{j=1}^{n} \mathfrak{c}_{j+1}^{\perp} \cap \mathfrak{c}_j = (\mathfrak{q}^{(0)} \cap \mathfrak{r}^{(0)}) \oplus \bigoplus_{j=1}^{n} (\mathfrak{q}^{(j)} \oplus \mathfrak{r}^{(j)}) \cap \mathfrak{q}^{(-j)} \cap \mathfrak{r}^{(-j)} \\
&= \bigoplus_{j=-n}^{n} \mathfrak{q}^{(j)} \cap \mathfrak{r}^{(-j)} = \bigoplus_{j=-n}^{n} \mathfrak{g}_j.
\end{aligned}$$

(1.15) tells us that $[\mathfrak{g}_j, \mathfrak{g}_k] \subset \mathfrak{g}_{j+k}$ where this makes sense, and so $\mathfrak{g}$ is a graded algebra. ∎

The map $\mathfrak{g}_j \mapsto j\mathfrak{g}_j$ is a derivation of $\mathfrak{g}$. Since all derivations of a semi-simple algebra are inner and $\operatorname{ad}\mathfrak{g} \cong \mathfrak{g}$ is an isomorphism there exists a unique $\xi \in \mathfrak{g}_0$ such that $(\operatorname{ad}\xi)|_{\mathfrak{g}_j} = j, \ \forall j$. ξ is the *grading* or *canonical element* of the pair $(\mathfrak{q}, \mathfrak{r})$. Notice that ξ both determines and is determined by the pair $(\mathfrak{q}, \mathfrak{r})$.

The existence of canonical elements yields an important result:

Lemma 1.12
Parabolic subalgebras are self-normalising.

Proof Let ξ be any canonical element for $\mathfrak{q}$. Then $[\xi, \mathfrak{q}^{\perp}] = \mathfrak{q}^{\perp} \Rightarrow [\mathfrak{q}, \mathfrak{q}^{\perp}] = \mathfrak{q}^{\perp}$ ((1.15) on its own only gives inclusion in one direction). Now $[\alpha, \mathfrak{q}] \subset \mathfrak{q} \iff B([\alpha, \mathfrak{q}], \mathfrak{q}^{\perp}) = 0 \iff B(\alpha, [\mathfrak{q}, \mathfrak{q}^{\perp}]) = 0 \iff \alpha \in [\mathfrak{q}, \mathfrak{q}^{\perp}]^{\perp} = \mathfrak{q}$. Thus the normaliser of $\mathfrak{q}$ is as claimed. ∎

Given a fixed parabolic subalgebra $\mathfrak{q}$, we know that complementary parabolic subalgebras exist. The next proposition tells us exactly how large the set $C_{\mathfrak{q}}$ of complementary parabolic subalgebras is.

Proposition 1.13
Given a fixed complementary pair $(\mathfrak{q}, \mathfrak{r})$, the map $n \mapsto \operatorname{Ad}\exp n\,\mathfrak{r}$ is a bijection $\mathfrak{q}^{\perp} \to C_{\mathfrak{q}}$.

Proof We prove this only for height 1 parabolic subalgebras. Let $(\mathfrak{q}, \mathfrak{r})$ be a fixed complementary pair with canonical element ξ. Let $X \in \mathfrak{q}$ and consider, for each j, the induced map (well-defined by (1.15))

$$\mathrm{ad}_j X : \mathfrak{q}^{(j)}/\mathfrak{q}^{(j+1)} \to \mathfrak{q}^{(j)}/\mathfrak{q}^{(j+1)} : Y \mapsto \mathrm{ad}(X)Y \pmod{\mathfrak{q}^{(j+1)}}.$$

It is clear that $\mathrm{tr}\,\mathrm{ad}\,X = \sum_{j=-n}^{n} \mathrm{tr}\,\mathrm{ad}_j X$. Furthermore $\mathfrak{q}^{(j)}/\mathfrak{q}^{(j+1)} = \mathfrak{g}_j + \mathfrak{q}^{(j+1)}$, $\forall j$. For $Y \in \mathfrak{q}^{(j)}$,

$$\mathrm{ad}_j \xi(Y + \mathfrak{q}^{(j+1)}) \equiv \mathrm{ad}\,\xi(Y) \equiv \mathrm{ad}\,\xi\left(Y|_{\mathfrak{g}_j}\right) \equiv j\, Y|_{\mathfrak{g}_j} \equiv jY \pmod{\mathfrak{q}^{(j+1)}},$$

which is independent of ξ and thus of the choice of complementary parabolic. Hence

$$B(X,\xi) = \mathrm{tr}\,\mathrm{ad}\,X\,\mathrm{ad}\,\xi = \sum_j \mathrm{tr}\,\mathrm{ad}_j X\,\mathrm{ad}_j \xi = \sum_j j\,\mathrm{tr}\,\mathrm{ad}_j X$$

is independent of ξ and so if ξ' is the grading element with respect to a second complement $\mathfrak{r}'$ of $\mathfrak{q}$ then $\xi - \xi' = n \in \mathfrak{q}^{\perp}$. Therefore

$$\mathrm{Ad}\exp(n)\xi = \xi + [n,\xi] = \xi - n = \xi'$$

and so $\mathfrak{r}' = \mathrm{Ad}\exp(n)\mathfrak{r}$. Conversely, suppose $n \in \mathfrak{q}^{\perp}$ and define $\mathfrak{r}' = \mathrm{Ad}\exp(n)\mathfrak{r}$. Since $\mathrm{Ad}\exp(n)$ preserves $\mathfrak{q}$ it is clear that $(\mathfrak{q},\mathfrak{r})$ are complimentary with canonical element $\xi' := \mathrm{Ad}\exp(n)\xi$.

∎

If $\mathrm{ht}(\mathfrak{q}) > 1$ the proposition still holds in that ξ', ξ are conjugate by a unique element $n \in \mathfrak{q}^{\perp}$. The construction of n is however a little more delicate than the above.

Proposition 1.13 says that complementarity is an open condition: putting the vector space topology of $\mathfrak{q}^{\perp}$ on $C_{\mathfrak{q}}$ makes the correspondence into an isomorphism of $\mathfrak{q}^{\perp}$ with a neighbourhood of $\mathfrak{r}$ in its G-conjugacy class, while one can reverse the analysis to see that there exists a neighbourhood $U \subset G$ of the identity such that $(\Phi \cdot \mathfrak{q}, \Phi \cdot \mathfrak{r})$ are complementary for any $\Phi \in U$. Indeed the conjugacy class of $\mathfrak{r}$ is a manifold[9] of which $C_{\mathfrak{q}}$ is a dense open subset, thus making the correspondence into a diffeomorphism.

While the existence of parabolic subalgebras of a complex semi-simple Lie algebra is guaranteed, the existence of parabolic subalgebras with Abelian nilradical is not. Indeed as [18, Theorem. 4.1] and [38, pg. 87, ex. 6] tell us, if $\mathfrak{h}$ is a Cartan subalgebra[10] of $\mathfrak{g}^{\mathbb{C}}$ with positive root system Δ^{+} and simple roots $\alpha_1, \dots, \alpha_l$ then any complex

[9] An R-space: see Section 5.3.

[10] Maximal semisimple Abelian subalgebra.

parabolic subalgebra is conjugate to a unique

$$\mathfrak{q}_I = \mathfrak{h} \oplus \sum_{n_I(\alpha) \geq 0} \mathfrak{g}^\alpha$$

where $\mathfrak{g}^\alpha = \{x \in \mathfrak{g}^{\mathbb{C}} : \mathrm{ad}(h)x = \alpha(h)x, \forall h \in \mathfrak{h}\}$ is the root space of α, $I \subset \{1, \dots, l\}$ is a multi-index and $n_I(\alpha) = \sum_{i \in I} n_i$ where $\alpha = \sum_{i=1}^l n_i \alpha_i$ is a height function. Furthermore the nilradical is conjugate to

$$\mathfrak{q}_I^\perp = \sum_{n_I(\alpha) > 0} \mathfrak{g}^\alpha.$$

Suppose for a moment that $\mathfrak{g}^{\mathbb{C}}$ is simple. It is not difficult to see that if the *weight*[11] of α_i is greater than 1, then $\mathfrak{q}_I^\perp$ is non-Abelian for $i \in I$. Furthermore, if $\alpha_i, \alpha_j \in \mathfrak{q}_I^\perp$, $i \neq j$ it is easy to see that $\mathfrak{q}_I^\perp$ is non-Abelian. Conversely $\sum_{n_{\{i\}}(\alpha)} \mathfrak{g}^\alpha$ is obviously Abelian if the weight of α_i is 1. It follows that there are as many distinct conjugacy classes of parabolic subalgebras of $\mathfrak{g}^{\mathbb{C}}$ as there are simple roots of weight 1 in the Dynkin diagram. In particular there do not exist height 1 parabolic subalgebras of the exceptional algebras $\mathfrak{e}_8, \mathfrak{f}_4, \mathfrak{g}_2$. For non-simple algebras, since root spaces corresponding to distinct irreducible subsystems commute, one may choose at most one simple root of weight 1 from each irreducible subsystem. Thus if the decomposition of $\mathfrak{g}^{\mathbb{C}}$ into simple ideals has n components, each of which has β_i simple roots of weight 1, then there are $\prod_{i=1}^n (\beta_i + 1) - 1$ distinct conjugacy classes of height 1 parabolic subalgebras of $\mathfrak{g}^{\mathbb{C}}$. As an example, every simple root in $\mathfrak{sl}(n+1, \mathbb{C})$ has weight 1 and so there exist n distinct conjugacy classes of parabolic subalgebras of height 1: these will be seen to correspond to the Grassmannians $G_k(\mathbb{R}^{n+1})$, $k = 1, \dots n$. This example, and other examples of conjugacy classes of parabolic subalgebras, will be discussed further in Section 5.3 when we introduce (symmetric) R-spaces. Conjugacy classes of real parabolic subalgebras are less easy to discuss in the abstract since one has a choice of real form of $\mathfrak{g}^{\mathbb{C}}$.

[11]The coefficient of α_i in the expansion of the highest root: note that the highest root space is automatically in the nilradical.

Chapter 2

Line Congruences and the Conformal Gauss Map

2.1 Introduction

A line congruence ℓ is an immersion of a real 2-manifold into the space of lines in real projective 3-space. In this chapter we show how studying the conformal geometry of a line congruence, viewed as a map into the Klein quadric $\mathbb{P}(\mathcal{L}^{3,3})$, provides a different understanding of the two classical invariants of a line congruence, those of Weingarten and Laplace. Indeed we will see that these invariants may be defined entirely in terms of the conformal invariants of ℓ and without reference to the existence of focal surfaces in $\mathbb{P}^3$ or special co-ordinates as their classical definitions require. We also observe that the Laplace transforms $\ell^{\pm}$ (defined by rotating ℓ throughout 90° with respect to the second fundamental form on each focal surface) have an invariant description as the null directions in the weightless normal bundle to ℓ. The dual congruence ℓ^* is introduced and we see that the Laplace and Weingarten invariants of the dual congruence are also related in a simple way to the conformal invariants of ℓ. In the finale we consider line congruences in terms of the structure equations of a natural moving frame of $\mathbb{R}^4$. In so doing we define isothermic line congruences (to be considered more fully in Chapter 6) and provide a proof of the Demoulin–Tzitzeica theorem [25, 58], which effectively says that isothermic line congruences have isothermic Laplace transforms. We also investigate when the Plücker image of a line congruence lies in a linear complex (the intersection of a hyperplane with the Klein quadric).

Definition 2.1
A line congruence $\ell : \Sigma \to G_2(\mathbb{R}^4)$ *is a map of a real 2-manifold* Σ *into the space of lines in* $\mathbb{P}^3$*. Usually* ℓ *is given as the join of a pair of distinct surfaces* $f, g : \Sigma \to \mathbb{P}^3$ *and we write* $\ell = f \wedge g$*.*

The fundamental object of interest to us will be the space of lines in $\mathbb{P}^3 = \mathbb{P}(\mathbb{R}^4)$, otherwise known as the Grassmannian $G_2(\mathbb{R}^4)$ of 2-planes in $\mathbb{R}^4$. Of all the Grassmannians, $G_2(\mathbb{R}^4)$ has perhaps been more extensively studied than any other because of two special properties. Firstly $G_2(\mathbb{R}^4)$ is a conformal manifold: the tangent bundle carries a natural family of conformal inner products. Indeed

$$T_\pi G_2(\mathbb{R}^4) \cong \hom(\pi, \mathbb{R}^4/\pi)$$

and so choosing bases on $\pi, \mathbb{R}^4/\pi$ and setting $(A, A) := \det(A)$ with respect to said bases defines a (2,2)-signature inner product. A different choice of bases simply scales the determinant and so all possible inner products so defined are conformal. The second special property is the Klein correspondence: a diffeomorphism between $G_2(\mathbb{R}^4)$ and the null lines in a (3,3)-signature vector space. It will be seen that the Klein correspondence is in fact a conformal diffeomorphism, for the conformal structure induced on $TG_2(\mathbb{R}^4)$ by the Klein correspondence is identical to that obtained above by determinants. In some ways this makes the first property an avatar of the second.

2.2 The Klein Correspondence

The Klein correspondence is a special case of the Plücker embedding of the Grassmannian $G_k(\mathbb{R}^n)$ in the projective space $\mathbb{P}^{\binom{n}{k}-1}$ (see e.g. [34, pg.209]). Uniquely among the Grassmannians, the Plücker map of $G_2(\mathbb{R}^4)$ is a bijection onto a quadric: specifically the Klein quadric in $\mathbb{P}^5$ defined by the vanishing of a (3,3)-signature quadratic form on $\mathbb{R}^6$. The bijective nature of the Plücker embedding of $G_2(\mathbb{R}^4)$ is a consequence of the group isomorphism $\mathrm{PSL}(4) \cong \mathrm{PSO}(3,3)$ via $A \mapsto \hat{A}$ where

$$\hat{A} : \mathsf{u} \wedge \mathsf{v} \mapsto A\mathsf{u} \wedge A\mathsf{v}. \tag{2.1}$$

We summarise the important points of the Klein correspondence, focusing on results required later: for a more detailed discussion see e.g. [62].

In order to describe a line ℓ in $\mathbb{P}^3$ we must specify the 2-dim subspace U of $\mathbb{R}^4$ such that $\ell = \mathbb{P}(U)$. Choose any two vectors u, v spanning U and abstractly associate to U the *bivector* $\mathsf{u} \wedge \mathsf{v}$. Denoting by $\bigwedge^2\mathbb{R}^4$ the linear space generated by all bivectors, *define* the map $\wedge : \mathbb{R}^4 \times \mathbb{R}^4 \to \bigwedge^2\mathbb{R}^4 : (\mathsf{u}, \mathsf{v}) \mapsto \mathsf{u} \wedge \mathsf{v}$ by insisting upon bilinearity and skew-symmetry. $\wedge$ is extended as a map $\bigwedge^2\mathbb{R}^4 \times \bigwedge^2\mathbb{R}^4 \to \bigwedge^4\mathbb{R}^4$ in the same fashion.[1] $\bigwedge^2\mathbb{R}^4$ is a 6-dim vector space (if $\{\mathsf{e}_i\}$ is a basis of $\mathbb{R}^4$ then $\{\mathsf{e}_i \wedge \mathsf{e}_j : i < j\}$ is a basis of $\bigwedge^2\mathbb{R}^4$). Now choose a second pair of vectors $\hat{\mathsf{u}} = \alpha\mathsf{u} + \beta\mathsf{v}$, $\hat{\mathsf{v}} = \gamma\mathsf{u} + \delta\mathsf{v}$ $(\alpha\delta - \beta\gamma \neq 0)$

[1] Note that $\wedge$ is *symmetric* over bivectors.

spanning U. Then

$$\hat{\mathsf{u}} \wedge \hat{\mathsf{v}} = (\alpha\delta - \beta\gamma)\mathsf{u} \wedge \mathsf{v}$$

and so the map $K : G_2(\mathbb{R}^4) \to \mathbb{P}(\bigwedge^2\mathbb{R}^4) : \langle \mathsf{u}, \mathsf{v}\rangle \mapsto \langle \mathsf{u} \wedge \mathsf{v}\rangle$ is well-defined. Conversely suppose $\langle \hat{\mathsf{u}} \wedge \hat{\mathsf{v}}\rangle = \langle \mathsf{u} \wedge \mathsf{v}\rangle$ and that $\hat{\mathsf{u}} \notin \langle \mathsf{u}, \mathsf{v}\rangle$. Choose a basis $\langle \mathsf{u}, \mathsf{v}, \hat{\mathsf{u}}, \mathsf{p}\rangle$ of $\mathbb{R}^4$ and write $\hat{\mathsf{v}} = a\mathsf{u} + b\mathsf{v} + c\hat{\mathsf{u}} + d\mathsf{p}$ to observe that

$$\hat{\mathsf{u}} \wedge \hat{\mathsf{v}} = a\hat{\mathsf{u}} \wedge \mathsf{u} + b\hat{\mathsf{u}} \wedge \mathsf{v} + d\hat{\mathsf{u}} \wedge \mathsf{p}.$$

Since $\hat{\mathsf{u}} \wedge \hat{\mathsf{v}} \in \langle \mathsf{u} \wedge \mathsf{v}\rangle$ and by linear independence of bases we have a contradiction and so $\hat{\mathsf{u}} \in U$. Similarly $\hat{\mathsf{v}} \in U$ and so the Klein correspondence injects.

A choice of *volume form* $\mathrm{vol} \in \left(\bigwedge^4\mathbb{R}^4\right)^*$ gives a symmetric quadratic form $\bigwedge^2\mathbb{R}^4 \times \bigwedge^2\mathbb{R}^4 \to \mathbb{R} : (\pi_1, \pi_2) \mapsto \mathrm{vol}(\pi_1 \wedge \pi_2)$ of signature (3,3) ($\mathsf{e}_i \wedge \mathsf{e}_j$ are null). Since $\bigwedge^4\mathbb{R}^4$ is a 1-dim space, any other choice of volume for simply scales the quadratic form. We therefore get a (3,3)-conformal structure on $\bigwedge^2\mathbb{R}^4$: consequently we will often refer to $\bigwedge^2\mathbb{R}^4$ as $\mathbb{R}^{3,3}$.

What about $\mathrm{Im}\, K$? Clearly $a \in \mathrm{Im}\, K \iff \exists \mathsf{u}, \mathsf{v} \in \mathbb{R}^4$ such that $a = \langle \mathsf{u} \wedge \mathsf{v}\rangle$. Such elements $\mathsf{u} \wedge \mathsf{v} \in \bigwedge^2\mathbb{R}^4$ are termed *decomposable*. Let $(\ ,\ )$ be a definite inner product on $\mathbb{R}^4$ and define $A \in \mathrm{End}(\mathbb{R}^4)$ by $\omega(\mathsf{u}, \mathsf{v}) = (A\mathsf{u}, \mathsf{v})$. Since A is skew, $\exists$ orthogonal basis $\{\mathsf{e}_i : |i| = 1, 2\}$ of $\mathbb{R}^4$ diagonalizing A: i.e. $\omega = \sum_{i=1,2} \lambda_i \mathsf{e}_i \wedge \mathsf{e}_{-i}$, for some scalars λ_i. Thus

$$\omega \wedge \omega = 2\lambda_1\lambda_2 \mathsf{e}_1 \wedge \mathsf{e}_{-1} \wedge \mathsf{e}_2 \wedge \mathsf{e}_{-2},$$

which is zero iff at most one of the λ_i is non-zero. Thus ω is decomposable iff $\omega \wedge \omega = 0$. Note that this result extends easily to $\bigwedge^2\mathbb{R}^n$. Decomposable elements are therefore exactly those elements of $\bigwedge^2\mathbb{R}^4$ which are null for the (3,3)-conformal structure defined above. Projectivising this result tells us that the space of lines in $\mathbb{P}^3$ is in bijective correspondence with the Klein quadric $\mathbb{P}(\mathcal{L}^{3,3})$, where $\mathcal{L}^{3,3}$ denotes the light-cone (null vectors) in $\mathbb{R}^{3,3}$.

The tangent bundle to $\mathbb{P}(\mathcal{L}^{3,3})$ is found by differentiating the nullity condition, i.e.

$$\begin{aligned} T_\ell\mathbb{P}(\mathcal{L}^{3,3}) &\cong \{A \in \hom(\ell, \mathbb{R}^{3,3}/\ell) : (As + \ell, s) = 0,\ \forall s \in \ell\} \\ &= \hom(\ell, \ell^\perp/\ell). \end{aligned} \tag{2.2}$$

which, up to tensoring by ℓ^*, is the (2,2)-signature quotient space $\ell^\perp/\ell$. Transferring an inner product to $\ell^\perp/\ell$ involves choosing a section of ℓ, any other choice scales the inner product: the Klein quadric is therefore a (2,2)-signature conformal manifold (actually a diffeomorph of $(S^2 \times S^2)/\mathbb{Z}_2$—see [1] for further details).

The conformality of the Klein quadric was to be expected since we already have a conformal structure on $G_2(\mathbb{R}^4)$: it should be checked that the two structures coincide. Let $\ell = \langle \mathsf{u}, \mathsf{v} \rangle$, $\mathbb{R}^4/\ell = \langle \mathsf{p}, \mathsf{q} \rangle \pmod{\ell}$: given $A \in \hom(\ell, \mathbb{R}^4/\ell)$ define $\hat{A} \in \hom(K\ell, (K\ell)^\perp/\ell)$ by differentiating (2.1)[2]

$$\hat{A}(\mathsf{u} \wedge \mathsf{v}) := A\mathsf{u} \wedge \mathsf{v} + \mathsf{u} \wedge A\mathsf{v} \pmod{\ell}.$$

It is easy to see that

$$(\hat{A}(\mathsf{u} \wedge \mathsf{v})) \wedge (\hat{A}(\mathsf{u} \wedge \mathsf{v})) = 2\det(A)\mathsf{p} \wedge \mathsf{v} \wedge \mathsf{u} \wedge \mathsf{q}$$

where det is calculated with respect to the given bases of ℓ and $\mathbb{R}^4/\ell$ and so the conformal structures really do line up.

Only one further consequence of the Klein correspondence will be required: a pair of lines ℓ_1, ℓ_2 in $\mathbb{P}^3$ intersect iff $\ell_1 \wedge \ell_2 = 0$. For this, notice that $\ell_1 = \mathsf{u}_1 \wedge \mathsf{v}_1, \ell_2 = \mathsf{u}_2 \wedge \mathsf{v}_2$ wedge to zero iff $\{\mathsf{u}_1, \mathsf{u}_2, \mathsf{v}_1, \mathsf{v}_2\}$ are linearly dependent. Then (say) $\mathsf{u}_1 = \alpha\mathsf{u}_2 + \beta\mathsf{v}_1 + \gamma\mathsf{v}_2 \Rightarrow \ell_1 = (\alpha\mathsf{u}_2 + \gamma\mathsf{v}_2) \wedge \mathsf{v}_1$. But $\alpha\mathsf{u}_2 + \gamma\mathsf{v}_2 \in \ell_2$ and so ℓ_1, ℓ_2 intersect.

It is worth taking a moment to clarify notation. The symbol ℓ will repeatedly refer to three different objects depending on context: a bundle of lines in $\mathbb{P}^3$ (2-planes in $\mathbb{R}^4$) over a base manifold Σ, the Plücker image of the same bundle in the Klein quadric $\mathbb{P}(\mathcal{L}^{3,3}) \subset \mathbb{P}^5$ and finally a global section (a *lift*) of this into the light-cone $\mathcal{L}^{3,3} \subset \mathbb{R}^{3,3}$. Notice that global sections exist for any light-cone $\mathcal{L}^{i,j}$ ($\max(i,j) > 1$): a smooth choice of either of the antipodal sections $\ell \cap S^{i+j}$, where S^{i+j} is a Euclidean unit sphere in $\mathbb{R}^{i+j}$, will do. It should be clear from the context which meaning we should ascribe to ℓ (in particular if $\ell = \ell(x,y)$ is a map from some subset of $\mathbb{R}^2$ then the partial derivative ℓ_x can only be taken in the context of ℓ being a lift into the light-cone). Points (on surfaces) in $\mathbb{P}^3$ will have a normal font while lifts of the same into $\mathbb{R}^4$ will use the same letter in a sans serif font. Spans will be denoted by $\langle\ \rangle$ (thus $\mathsf{f} \in f \iff f = \langle \mathsf{f} \rangle$) and inner products by $(\ ,\ )$. The notation $\ell = f \wedge g$ for the join of two points in $\mathbb{P}^3$ (as used in Definition 2.1) is now seen to be motivated by the Klein correspondence and will hopefully cause no confusion with the Plücker image of ℓ: i.e. $\ell = f \wedge g = \langle \mathsf{f} \wedge \mathsf{g} \rangle \in \mathbb{P}(\mathcal{L}^{3,3})$.

[2] From now on we drop the K when referring to the Plücker image of ℓ.

2.3 Focal Surfaces and Laplace Transforms

We focus on some of the classical theory of line congruences in $\mathbb{P}^3$, albeit in a more modern setting. In particular we describe a line congruence in terms of its focal surfaces.

Definition 2.2
A focal surface *of a line congruence ℓ is a surface $f : \Sigma \to \mathbb{P}^3$ with which ℓ has first-order contact: that is*

$$\left.\begin{array}{l} f(p) \in \ell(p) \\ \ell(p) \leq T_{f(p)}f \end{array}\right\}, \forall p \in \Sigma. \tag{2.3}$$

Equivalently $\ell \in \Gamma(\mathbb{P}(Tf)) = \{\text{sections of the projective tangent bundle of } f\}$.

Proposition 2.3
A generic line congruence has two complex focal surfaces.

Proof Let $f : \Sigma \to \mathbb{P}^3$ intersect ℓ. The tangent space $T_{f(p)}\mathbb{P}^3$ is $\hom(f(p), \mathbb{R}^4/f(p))$ which, up to tensoring by a line bundle, is $\mathbb{R}^4/f(p)$. Therefore[3] $Tf \cong \langle \mathsf{f}, \mathsf{df}\rangle \;/\; \langle \mathsf{f}\rangle$, independent of choice of lift. f is therefore focal for ℓ iff $\ell \leq \langle \mathsf{f}, \mathsf{df}(T_p\Sigma)\rangle \iff \ell \wedge \mathsf{df}(T_p\Sigma) = 0$ (i.e. the 2-planes ℓ and $\mathsf{df}(T_p\Sigma)$ intersect). If $\ell = u \wedge v$ is given in terms of any two surfaces intersecting ℓ, then $\mathsf{f} = \alpha\mathsf{u} + \beta\mathsf{v}$ for some α, β where $[\alpha, \beta]$ are the homogeneous co-ordinates of a point in $\mathbb{P}^1$. Choose local co-ordinates x, y on Σ, then

$$\ell \wedge \mathsf{df}(T_p\Sigma) = 0 \iff \mathsf{u} \wedge \mathsf{v} \wedge (\alpha\mathsf{u}_x + \beta\mathsf{v}_x) \wedge (\alpha\mathsf{u}_y + \beta\mathsf{v}_y) = 0. \tag{2.4}$$

Since $\bigwedge^4\mathbb{R}^4 \cong \mathbb{R}$ this is a quadratic condition on α, β for which there are generically two solutions for $[\alpha, \beta]$.

∎

All outcomes of the quadratic condition (2.4) are possible: a congruence lying entirely in a plane will have f focal for all α, β while the lengthways tangent lines to a cylinder have only one focal surface (the cylinder itself). A congruence is *non-degenerate* if it has two distinct focal surfaces. We will tend to define congruences by specifying a single real focal surface and a tangent direction. The second focal surface is then manifestly real by the quadratic condition. We will therefore restrict attention to non-degenerate congruences with real focal surfaces. With minor modifications the following analysis still holds when ℓ has complex conjugate focal surfaces. The main

[3] By df we mean the image $\mathsf{df}(T\Sigma)$.

difference will be seen to be in the signature of the conformal structure on Σ induced by the Klein quadric: real focal surfaces induce a (1,1)-signature while complex conjugate focal surfaces induce a definite structure. This will be discussed more fully in Section 2.5.

If we were to take our cue from Eisenhart ([28, pg.10]), we would have insisted upon all congruences being non-degenerate. A *developable surface* is a 1-parameter family of lines which is tangent to a curve, or such that all lines meet in a point (a 0-dimensional curve). A non-degenerate congruence has exactly two such developable surfaces through each line in the congruence. It is easy to see that a subfamily of $\ell = u \wedge v$ is developable iff $\exists$ co-ordinate x on Σ such that $\mathsf{u} \wedge \mathsf{v} \wedge \mathsf{u}_x \wedge \mathsf{v}_x = 0$ (since a curve $\mathsf{f} = \alpha\mathsf{u} + \beta\mathsf{v}$ has $\mathsf{f} \wedge \mathsf{f}_x \in u \wedge v \iff \alpha\mathsf{u}_x + \beta\mathsf{v}_x \in u \wedge v$). There are therefore exactly as many developable surfaces through any line in ℓ as there are solutions to (2.4). Eisenhart defines a line congruence in an arbitrary dimensional vector space to be a 2-parameter family of lines for which there exist two developable surfaces of the congruence through each line. In $\mathbb{P}^3$, as we have seen, this is generic, but in higher dimensions this is a strong condition. We will consider congruences in higher dimensions in Chapter 6.

Let $\ell : \Sigma \to G_2(\mathbb{R}^4)$ be a non-degenerate line congruence with real immersed focal surfaces $f, g : \Sigma \to \mathbb{P}^3$. ℓ is tangent to both f, g and so picks out a direction on each focal surface and, since f, g are immersions, picks out two directions on Σ. This decomposes the tangent bundle to Σ into two line bundles

$$T\Sigma = L_- \oplus L_+, \tag{2.5}$$

defined by $\mathrm{d}f(L_+), \mathrm{d}g(L_-) \subset \ell$. For the present we ignore the possibility that $L_- = L_+$: it will be seen in Section 2.5 that $L_- = L_+$ corresponds to a degenerate congruence with only one focal surface. Let $X \in L_-$, $Y \in L_+$. Frobenius' theorem (e.g. [43, 54, 63]) tells us that there exist local co-ordinates x, x_1 on Σ such that $X = \frac{\partial}{\partial x_1}$ and the integral curves of L_- are given by $x =$ constant ($\mathrm{d}x(L_-) = 0$). Similarly $\exists y, y_1$ such that $Y = \frac{\partial}{\partial y_1}$ and $\mathrm{d}y(L_+) = 0$. Thus $T^*\Sigma = \langle \mathrm{d}x, \mathrm{d}y \rangle$ and we have constructed a local co-ordinate system on Σ such that

$$\ell = \langle \mathsf{f} \wedge \mathsf{g} \rangle = \langle \mathsf{f} \wedge \mathsf{f}_y \rangle = \langle \mathsf{g} \wedge \mathsf{g}_x \rangle . \tag{2.6}$$

x, y are *conformal co-ordinates* for ℓ and are defined only up to scaling: $u(x)$, $v(y)$ are also conformal for any monotone, non-zero functions u, v. *Conformal* refers to the fact that ℓ_x, ℓ_y are the null directions of the conformal structure on $\operatorname{Im} \mathrm{d}\ell$ induced by the Klein quadric (see Section 2.5).

The co-ordinates x, y have a different flavour when we look at focal surfaces. By (2.6) there exist scalar functions a, b, d, h such that

$$\mathsf{f}_y = -a\mathsf{f} + d\mathsf{g}, \quad \mathsf{g}_x = h\mathsf{f} - b\mathsf{g}. \tag{2.7}$$

By taking f as fundamental, we *define* a lift of g by fixing $d = 1$. Consequently f, g satisfy a *Laplace equation*:

$$\begin{aligned} &\mathsf{f}_{xy} + a\mathsf{f}_x + b\mathsf{f}_y + c\mathsf{f} = 0, \\ &\mathsf{g}_{xy} + a_1\mathsf{g}_x + b_1\mathsf{g}_y + c_1\mathsf{g} = 0, \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} &c := ab + a_x - h, \qquad a_1 := a - (\ln h)_y, \qquad b_1 := b, \\ &c_1 := ab + b_y - h - b(\ln h)_y = c + b_y - a_x - b(\ln h)_y. \end{aligned} \tag{2.9}$$

Our choice of notation is borrowed from [51] and we single out $h = ab - c + a_x$ due to the starring role it will play in Section 2.4. Consider the second fundamental forms of the focal surfaces. In projective space the concept of a unit normal vector to a surface is not defined, however we can canonically define the *dual surface* to a surface which plays much the same role.

Definition 2.4
Given an immersed surface $f : \Sigma \to \mathbb{P}^3$, the dual surface *$f^* : \Sigma \to \mathbb{P}^3_* = \mathbb{P}(\mathbb{R}^4_*)$ is the annihilator of $\langle f, \mathrm{d}f \rangle$.*

Equivalently f^* is the span of the linear operator $\mathsf{p} \mapsto \mathrm{vol}(\mathsf{f} \wedge \mathsf{f}_x \wedge \mathsf{f}_y \wedge \mathsf{p})$ for *any* lifts f, p and any co-ordinates x, y on Σ. Instead of performing calculations with this linear operator we will later (Section 2.8) define a canonical lift f^* of f^* in another way.

Definition 2.5
The second fundamental form *$\mathrm{I\!I}_f$ of a surface in $\mathbb{P}^3$ is the conformal class of the metrics $\mathsf{f}^*(\mathsf{f}_{xx}\,\mathrm{d}x^2 + 2\mathsf{f}_{xy}\,\mathrm{d}x\,\mathrm{d}y + \mathsf{f}_{yy}\mathrm{d}y^2)$.*

It is easy to see that different choices of lifts of f, f^* preserve the conformal class and so $\mathrm{I\!I}_f$ really is an invariant of the surface.

(2.8) now says that x, y are *conjugate* co-ordinates ($\mathrm{I\!I}(\partial_x, \partial_y) = 0$) for both focal surfaces f, g. Suppose that $\ell = f \wedge g$ is non-degenerate, then $\mathsf{f}, \mathsf{g}, \mathsf{f}_x, \mathsf{g}_y$ are linearly independent (if not then every 4-form in (2.4) vanishes), so introduce functions δ, γ' :

$\Sigma \to \mathbb{R}$ such than

$$\begin{aligned} \mathsf{f}_{xx} &\equiv \delta \mathsf{g}_y \pmod{\mathsf{f}, \mathsf{g}, \mathsf{f}_x}, \\ \mathsf{g}_{yy} &= \gamma' \mathsf{f}_x \pmod{\mathsf{f}, \mathsf{g}, \mathsf{g}_y}. \end{aligned} \tag{2.10}$$

The second fundamental forms are the conformal classes

$$\mathbb{I}_f = \langle \delta \, dx^2 + dy^2 \rangle, \qquad \mathbb{I}_g = \langle h \, dx^2 + \gamma' \, dy^2 \rangle. \tag{2.11}$$

It should be noted that the imposition of a conjugate net on a focal surface is a piece of additional structure—a immersed surface in $\mathbb{P}^3$ has an infinity of conjugate nets (just pick a direction and rotate 90^o with respect to $\mathbb{I}$ for the other direction). Such a set-up fails in higher dimensions when there is no second fundamental form. It also fails when we consider asymptotic congruences—the family of lines tangent to the asymptotic directions on f, i.e. $\mathsf{f} \wedge \mathsf{f}_z$ where ∂_z is a null direction of $\mathbb{I}$. If $\bar{z}$ is the other asymptotic co-ordinate, then the 4-form (2.4) applied to $\lambda \mathsf{f} + \mu \mathsf{f}_z$ becomes

$$\mu \mathsf{f} \wedge \mathsf{f}_z \wedge \mathsf{f}_{\bar{z}} \wedge \mathsf{f}_{z\bar{z}},$$

which is either always zero, in which case $\mathbb{I}_f \equiv 0$ and any surface intersecting $f \wedge f_z$ is focal, or never zero, whence f is the only focal surface. Asymptotic congruences are therefore always degenerate.

So far we have used only one co-ordinate on each focal surface to define a congruence. Define

$$\ell^{-1} = \langle \mathsf{f} \wedge \mathsf{f}_x \rangle, \qquad \ell^1 = \langle \mathsf{g} \wedge \mathsf{g}_y \rangle.$$

It is easy to see that second focal surfaces of these new congruences are $f^{-1} := \langle \mathsf{f}_x + b\mathsf{f} \rangle$ and $g^1 := \langle \mathsf{g}_y + a_1 \mathsf{g} \rangle$ respectively.

Definition 2.6
*f^{-1} is the $-ve$-*first Laplace transform *of f while $g = f^1$ is the first Laplace transform. Similarly ℓ^{-1}, ℓ^1 are Laplace transforms of ℓ.*

It is an easy, if tedious, exercise to check that Laplace transforms are well-defined up to the freedoms inherent in our set-up, namely the choice of lift f and the choice of conformal co-ordinates. The functions a, b, c, h will change, but ℓ^{-1}, ℓ^1 and so f^{-1}, g^1 are well-defined. When we write f^{-1} for a lift of f^{-1} it is to be understood that $\mathsf{f}^{-1} = \mathsf{f}_x + b\mathsf{f}$ is given in terms of our choice of f and co-ordinates x, y.

We now have a well-defined recipe for finding new line congruences from old: given a line congruence ℓ find a focal surface g, rotate 90° with respect to II_g to define a new congruence ℓ^1, find the second focal surface of ℓ^1, etc. This *Laplace sequence* makes sense as long as focal surfaces and congruences do not degenerate. Such a sequence was originally studied in order to investigate the transformation of hyperbolic PDEs (2.8) for scalar f, g. Since Laplace transforms of solutions satisfy a new PDE with different coefficients it was hoped that a known equation might appear somewhere in the sequence. Solving the original equation is then simply a matter of keeping track of the coefficients a, b, c using iteration formulae (2.9).

Notice that if ℓ is non-degenerate then $\ell = Tf \cap Tg$ and f, g, f^{-1}, g^1 are independent. Upon fixing a lift of f we have canonical choices for lifts of g, f^{-1}, g^1 and so a moving frame of $\mathbb{R}^4$. For future use define functions $\alpha, \dots, \delta' : \Sigma \to \mathbb{R}$ by

$$\begin{aligned} \mathrm{f}^{-1}_x &= \alpha\mathrm{f} + \beta\mathrm{g} + \gamma\mathrm{f}^{-1} + \delta\mathrm{g}^1, \\ \mathrm{g}^1_y &= \alpha'\mathrm{f} + \beta'\mathrm{g} + \gamma'\mathrm{f}^{-1} + \delta'\mathrm{g}^1 \end{aligned} \tag{2.12}$$

and observe that δ and γ' are exactly as defined in (2.10). We will work with the frame $\mathrm{f}, \mathrm{g}, \mathrm{f}^{-1}\mathrm{g}^1$ repeatedly in this chapter and will calculate its structure equations in Section 2.9.

2.4 The Laplace and Weingarten Invariants

As already discussed, we have made two choices in determining the Laplace sequence of a focal surface. Let us see how these choices affect the functions a, b, c. Under a change of lift $\mathrm{f} \mapsto \mu^{-1}\mathrm{f}$ the scalars a, b, c transform as

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} a + (\ln\mu)_y \\ b + (\ln\mu)_x \\ c + a(\ln\mu)_x + b(\ln\mu)_y + \mu_{xy}/\mu \end{pmatrix}.$$

Similarly, under the change of co-ordinates $u = u(x),\ v = v(y)$, we have

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} a\frac{\mathrm{d}y}{\mathrm{d}v} \\ b\frac{\mathrm{d}x}{\mathrm{d}u} \\ c\frac{\mathrm{d}x}{\mathrm{d}u}\frac{\mathrm{d}y}{\mathrm{d}v} \end{pmatrix}.$$

If we now set

$$h = ab - c + a_x, \qquad k = ab - c + b_y,$$

we see that the symmetric 2-forms

$$\mathcal{H} := h\,\mathrm{d}x\mathrm{d}y, \qquad \mathcal{K} := k\,\mathrm{d}x\mathrm{d}y,$$

are invariant under the above choices. $\mathcal{H}, \mathcal{K}$ are the *Laplace invariants* of the surface f with conjugate net x, y.[4] Repeating the calculations for g, f^{-1} gives

$$\mathcal{K}_g = \mathcal{H}_f, \text{ and } \mathcal{K}_f = \mathcal{H}_{f^{-1}}.$$

It would therefore seem more logical to dispense with the classical identification and label the Laplace invariants as belonging to line congruences rather than focal surfaces. Accordingly we use the notation $\mathcal{H}_i,\ i \in \mathbb{Z}$ to denote the Laplace invariant of ℓ^i. It should now be clear that the Laplace invariant of ℓ^i is independent of the order of the co-ordinates x, y: in case of confusion, $\mathcal{H}$ is calculated on either focal surface by taking $ab - c$ (for that surface) and adding a_x if y is the co-ordinate taking you to the other focal surface and b_y if x takes you to the new surface. In Section 2.7, $\mathcal{H}$ will be seen to depend directly on ℓ in another way, without having to work with focal surfaces. The following proposition is a simple matter of calculation.

Proposition 2.7
For as long as the Laplace sequence remains non-degenerate, the functions a_i, b_i, c_i satisfy

$$\begin{aligned}
&a_{i+1} = a_i - (\ln h_i)_y, \qquad b_{i+1} = b_i, && \multirow{2}{*}{} i \geq 0,\\
&c_{i+1} = c_i - (a_i)_x + (b_i)_y - b_i(\ln h_i)_y, && \\
&a_{i-1} = a_i, \qquad b_{i-1} = b_i - (\ln k_{|i|})_x, && i \leq 0,\\
&c_{i-1} = c_i + (a_i)_x - (b_i)_y - a_i(\ln k_{|i|})_x, &&
\end{aligned} \tag{2.13}$$

while the Laplace invariants satisfy

$$\begin{aligned}
&h_{i+1} + h_{i-1} = 2h_i - (\ln h_i)_{xy}, && \forall i,\\
&h_{i+1} = h_i + h_1 - h_0 - (\ln(h_1 \cdots h_i))_{xy}, && i > 0,\\
&k_{i+1} = k_i + k_1 - k_0 - (\ln(k_1 \cdots k_i))_{xy}. &&
\end{aligned} \tag{2.14}$$

A second classical invariant may be associated to a line congruence. The *Weingarten invariant* measures the difference between the conformal structures of the second fundamental forms on the focal surfaces of a congruence.

$$\mathcal{W} := W\,\mathrm{d}x\mathrm{d}y = (h - \delta\gamma')\,\mathrm{d}x\mathrm{d}y.$$

[4]Eisenhart [28] calls $\mathcal{H}, \mathcal{K}$ the *point invariants* of the conjugate net.

In exactly the same manner as for the Laplace invariants, $\mathcal{W}$ is invariant under choices of lifts and conjugate co-ordinates. It is immediate from (2.11) that $W = 0 \iff \mathbb{I}_f, \mathbb{I}_g$ are the same conformal class. Congruences with $W = 0$ are *W-congruences.* It is possible to obtain recurrence relations between successive Weingarten invariants, but we shall leave this until Section 2.8 when we have a better understanding of the geometric meaning of the invariant $\delta\gamma' \mathrm{d}x\mathrm{d}y$.

Now that all the classical machinery is in place, it is worth recording an example of Laplace transforms in action. For this we lift shamelessly from [51]. Consider the quadratic surface $XY = Z^2 + U^2$ in $\mathbb{P}^3$. Depending on the choice of affine chart this is an ellipsoid, a paraboloid or a hyperboloid in $\mathbb{R}^3$: e.g. $U = 1$ gives a 2-sheeted hyperboloid whilst $X = 1$ yields a paraboloid. The following parameterisation is in terms of conjugate co-ordinates:

$$f(x,y) = [\mathsf{f}(x,y)] = [e^{x+y}, e^{-x-y}, \cos(x-y), \sin(x-y)].$$

By direct computation we see that $\mathsf{f}_{xy} = \mathsf{f}$, hence $a = b = 0$, $c = -1$ and so $h = k = 1$. Furthermore

$$\begin{aligned} &\mathsf{f}^1 = \mathsf{g} = \mathsf{f}_y, && \mathsf{f}^2 = \mathsf{g}^1 = \mathsf{f}_{yy} + a_1\mathsf{g} = \mathsf{f}_{yy} = \mathsf{f}_{xx}, \\ &\mathsf{f}^3 = \mathsf{f}_{xxy} = \mathsf{f}_x, && \mathsf{f}^4 = \mathsf{f}_{xy} = \mathsf{f}. \end{aligned}$$

The Laplace sequence of f in these co-ordinates is therefore 4-periodic, with all focal surfaces being quadratic. The line congruences $\ell^i = f^i \wedge f^{i+1}$ have Laplace invariants $\mathcal{H}^i = \mathrm{d}x\mathrm{d}y$ and all are W-congruences. Figure 2-1 comprises two different views of segments of the four focal surfaces in the Laplace sequence joined by lines from the congruences and is obtained by taking the affine chart $X = 1$ on $\mathbb{P}^3$ so that

$$f(x,y) = (e^{-2(x+y)}, e^{-(x+y)}\cos(x-y), e^{-(x+y)}\sin(x-y))$$

is a paraboloid.

2.5 Co-ordinates and the Contact Structure

It is worth having a careful discussion about the conformal/conjugate co-ordinates x, y since they play several different roles. We need not work with focal surfaces at all to determine the existence of x, y. Indeed the following analysis applies equally to any immersion of a 2-manifold into a projective light cone $\mathbb{P}(\mathcal{L}^{i,j})$ for which the induced conformal structure is non-degenerate.

Given a general immersion $\ell : \Sigma \to \mathbb{P}(\mathcal{L}^{i,j})$, there are four possibilities for the

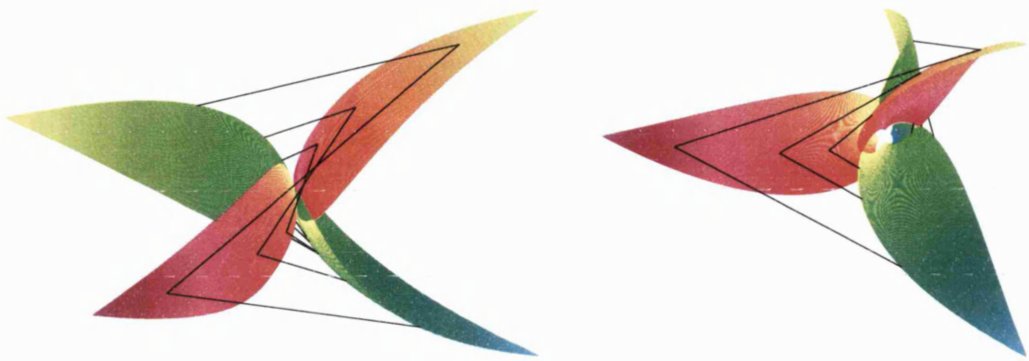

Figure 2-1: The Laplace sequence for f

induced conformal structure $(\mathrm{d}\ell, \mathrm{d}\ell)$ on Σ:

- $\operatorname{Im} \mathrm{d}\ell$ is isotropic: any α, β solve (2.4) and so there are a $\mathbb{P}^1$-worth of focal surfaces and the image under $\mathrm{d}\ell$ of any line subbundle $L \subset T\Sigma$ is null.
- $\operatorname{Im} \mathrm{d}\ell$ contains a single null line: (2.5) degenerates so that $L_- = L_+$ and there is only one focal surface.
- $\operatorname{Im} \mathrm{d}\ell$ is a (1,1)-plane: we are in the case of (2.5) where L_-, L_+ are real and distinct.
- $\operatorname{Im} \mathrm{d}\ell$ is definite: the null directions in $\operatorname{Im} \mathrm{d}\ell$ are complex conjugates and so we have a splitting $T^{\mathbb{C}}\Sigma = L \oplus \overline{L}$.

The first two cases describe degenerate congruences, while the final two describe real and complex conjugate focal surfaces respectively. The difference in the signature of $\operatorname{Im} \mathrm{d}\ell$ is one of the reasons that we do not consider complex focal surfaces in this account. The more obvious reason is that the Laplace transforms of such congruences are no longer real. The following theory remains completely true however, with slight modifications of signatures and co-ordinates when required.

Definition 2.8
Given a non-degenerate immersion $\ell : \Sigma^2 \to \mathbb{P}(\mathcal{L}^{i,j})$, conformal co-ordinates *for ℓ are local co-ordinates x, y on Σ satisfying*

$$(\ell_x, \ell_x) = 0 = (\ell_y, \ell_y), \qquad (\ell_x, \ell_y) \neq 0.$$

If $\operatorname{Im} \mathrm{d}\ell$ has signature (1,1) we can pull-back the null directions of the conformal structure to Σ, from which the Frobenius argument of Section 2.3 yields conformal co-ordinates. If ℓ has definite structure one can obtain complex conjugate conformal co-ordinates by appealing to the existence of isothermal co-ordinates [23]: the null directions of the conformal structure splits $T^{\mathbb{C}}\Sigma = L \oplus \overline{L}$ into complex conjugate line subbundles; there is a complex co-ordinate z on Σ such that $L = \left\langle \frac{\partial}{\partial z} \right\rangle$, $\overline{L} = \left\langle \frac{\partial}{\partial \bar{z}} \right\rangle$. Σ is therefore a Riemann surface. Observe that ℓ being either holomorphic or anti-holomorphic forces the degeneracy of ℓ, since $(\ell_z, \ell_{\bar{z}}) = 0$ contradicts Definition 2.8.

It is clear from the above that knowledge of focal surfaces is not required in order to obtain conformal co-ordinates: the conformal structure on $\operatorname{Im} \mathrm{d}\ell$ is enough to determine the line bundles $L_\pm$ and thus x, y (or $L, \overline{L}$ and z).

Contact structure

There are two further conformal structures on Σ induced by ℓ: those of the second fundamental forms of the focal surfaces f, g. We have already seen (2.11) that conformal co-ordinates x, y are conjugate for both $\mathbb{I}_f, \mathbb{I}_g$. There is another way of viewing $\mathbb{I}_f, \mathbb{I}_g$, which is more akin to the determinant construction on $T_\ell G_2(\mathbb{R}^4)$. Indeed $\mathbb{I}_f, \mathbb{I}_g$ can be recovered by the following discussion even when one refuses to work with focal surfaces.

Definition 2.9
A contact element *in $\mathbb{P}^3$ is a point-plane pair where the point lies on the plane. Equivalently it is the set of lines through a point in $\mathbb{P}^3$ which also lie in a plane. The set of contact elements Z is a* contact manifold. *The* contact lift *of a surface $f : \Sigma \to \mathbb{P}^3$ is a map $F : \Sigma \to Z$ where each point of f in $\mathbb{P}^3$ is replaced by the contact element tangent to f at that point.*

Throughout this section we use notation and analysis similar to that in [13]. For further discussion of contact manifolds and structures see [2, 5, 22].

Via the Klein correspondence, we know that the manifold Z of contact elements of $\mathbb{P}^3$ is the space of null-lines in the Klein quadric, or equivalently the Grassmannian of null 2-planes in $\mathbb{R}^{3,3}$. Either way the contact lift of a focal surface f is a map $F : \Sigma \to Z$ and, by considering the relation $\ell = \langle \mathsf{f} \wedge \mathsf{g} \rangle = Tf \cap Tg$, we see that the exact correspondence between the pictures in $\mathbb{P}^3$ and the Klein quadric is $\ell = F \cap G$ as in figure 2-2.

It is clear that $\mathrm{O}(3,3)$ acts transitively on Z and so we can view Z as a homogeneous $\mathrm{O}(3,3)$-space. Let the stabiliser of a fixed point $F_0 \in Z$ be $K \subset \mathrm{O}(3,3)$ so that $Z \cong G/K$. If $\mathfrak{k}$ is the Lie algebra of K then the tangent bundle is given by the usual identification $T_{gK}G/K \cong \mathrm{Ad}(g)(\mathfrak{g}/\mathfrak{k}) = \mathfrak{g}/\operatorname{stab}(g \cdot \mathfrak{k})$ of Section 1.1

$$\mathrm{Ad}(g)(\mathfrak{g}/\mathfrak{k}) \ni \xi + \mathrm{Ad}(g)\mathfrak{k} \mapsto \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(t\xi)gK. \tag{2.15}$$

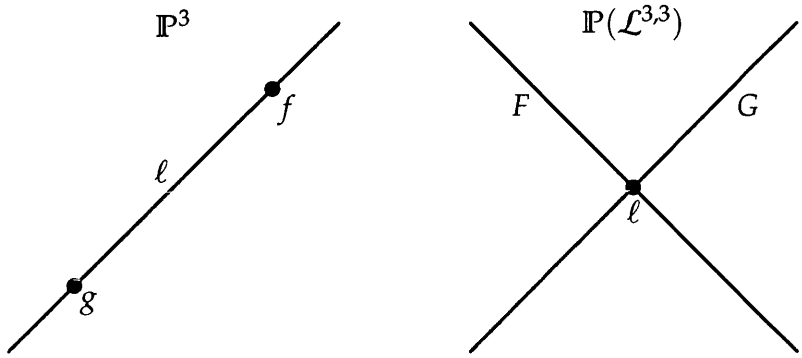

Figure 2-2: Contact lifts of surfaces in $\mathbb{P}^3$

That is

$$T_F Z \cong \mathfrak{so}(3,3)/\operatorname{stab}(F) = \{A \in \hom(F, \mathbb{R}^{3,3}/F) \mid (Ap_1, p_2) + (p_1, Ap_2) = 0\}$$

via $X \mapsto A_X$ where $A_X\sigma = \mathrm{d}_X\sigma \pmod F$. It is not difficult to see that $\dim Z = 5$ (in general the manifold of null 2-planes in $\mathbb{R}^{p,q}$, $p, q \geq 2$ has dimension[5] $2(p+q) - 7$).

Definition 2.10
A contact structure $\mathcal{D}$ *on an odd dimensional manifold* Z *is a codimension 1 subbundle* $\mathcal{D} \subset TZ$ *such that the map* $P : \bigwedge^2 \mathcal{D} \to TZ/\mathcal{D} : X, Y \mapsto \pi([X, Y])$ *is non-degenerate, where* $\pi : TZ \to TZ/\mathcal{D}$ *is the bundle projection.*

Remark: Writing L for the line bundle $TZ/\mathcal{D}$ we see that $P = -\mathrm{d}\pi$ on any patch where L is trivial. Our definition ties in with the more usual definition given in (e.g.) [7, 8, 65] where π is a *contact form* and $\mathcal{D}$ the *contact distribution*.

Z gets a contact structure by letting $\mathcal{D}$ be the subbundle of TZ with fibres

$$\mathcal{D}_F = \hom(F, F^\perp/F).$$

The map P is non-degenerate iff for every $A \in \mathcal{D}_F$ there exists a $B \in \mathcal{D}_F$ such that $\pi([A, B]) \neq 0$ which means there exist $f_1, f_2 \in \Gamma F$ such that $([A, B]f_1, f_2) \neq 0$. Given $A, B \in \mathcal{D}_F$ we extend by (2.15) to sections of $\mathcal{D} \subset TZ$:

$$\tilde{A}_p = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp tA \cdot p,$$

[5] $\dim Z = \dim \operatorname{stab}(F)^\perp = \dim(F^\perp \wedge F)$ via the isomorphism (3.7). Indeed $\operatorname{stab}(F)$ is a bundle of height 2 parabolic subalgebras of $\mathfrak{so}(3,3)$ (Section 1.3) and so Z is a height 2 R-space (Section 5.3).

while [37] (pg. 122) gives the relation $[\tilde{A}, \tilde{B}] = -\widetilde{[A,B]}$. Thus

$$\begin{aligned}(\widetilde{[A,B]}f_1, f_2) &= -([\tilde{A}, \tilde{B}]f_1, f_2) = (\tilde{B}\tilde{A}f_1, f_2) - (\tilde{A}\tilde{B}f_1, f_2)\\ &= (Bf_1, Af_2) - (Af_1, Bf_2).\end{aligned}$$

Fix A. Since we have a free choice of $Bf_1, Bf_2 \in F^\perp$ it is easy to find a B such that $(\widetilde{[A,B]}f_1, f_2) \neq 0$. We thus have non-degeneracy and so $\mathcal{D}$ is a contact structure.

In the case at hand when F, G are contact lifts of focal surfaces, both F and $F^\perp/F$ are 2-dimensional and so $\mathcal{D}$ gets a natural (2,2)-conformal structure from $(A, A) = \det A$. Choice of bases on $F, F^\perp/F$ are the only freedoms left to us, but these just scale the resulting determinant.

Definition 2.11

A map $F : \Sigma \to Z$ is Legendre *if it is tangent to the contact structure $\mathcal{D}$: i.e. if u, v are any sections of $F \to \Sigma$ then $(\mathrm{d}u, v) \equiv 0$.*

The contact lifts F, G are clearly Legendre (take $u = \ell$, $v = \ell^{-1}$, etc.). Since F is an immersion and $\operatorname{Im} \mathrm{d}F \subset \mathcal{D}$, we can pull back the conformal structure on $\mathcal{D}$ to Σ. The induced structure turns out to be exactly that of the second fundamental form of f. To see this, note that the null directions of II_f are the asymptotic lines, spanned by

$$v^\pm = \frac{\partial}{\partial x} \pm \sqrt{-\delta}\frac{\partial}{\partial y}.$$

Considering the maps

$$A_x : \sigma \mapsto \mathrm{Proj}^\perp(\sigma_x), \qquad A_y : \sigma \mapsto \mathrm{Proj}^\perp(\sigma_y),$$

where σ is a local section of F and $\mathrm{Proj}^\perp$ is projection in $F^\perp$ away from F, we see that,

$$A_{v^\pm} : \begin{pmatrix} \mathsf{f} \wedge \mathsf{g} \\ \mathsf{f} \wedge \mathsf{f}_x \end{pmatrix} \mapsto \begin{pmatrix} -1 & \pm\sqrt{-\delta} \\ \pm\sqrt{-\delta} & \delta \end{pmatrix} \begin{pmatrix} \mathsf{g} \wedge \mathsf{f}_x \\ \mathsf{f} \wedge \mathsf{g}_y \end{pmatrix} \quad \mathrm{mod}\ F$$

and so $\det A_{v^\pm} = 0$. The conformal structures have the same null directions and are thus identical.

We now have three distinct conformal structures on Σ induced by ℓ, the metric $(\mathrm{d}\ell, \mathrm{d}\ell)$ and the second fundamental forms of the focal surfaces $\mathrm{II}_f, \mathrm{II}_g$, each of which can be defined in two separate ways. The two second fundamental forms of course coincide when ℓ is a W-congruence.

The above analysis is not restricted to $\mathbb{P}(\mathcal{L}^{3,3})$. Indeed one may let Z be the set of null 2-planes in $\mathbb{R}^{4,2}$ and observe that Z is a homogeneous $O(4,2)$-space and a contact manifold. Since $4+2=6$ we again get a conformal structure of signature (2,2) on $\mathcal{D}$, which Legendre immersions will pull back to Σ. Given a line congruence with induced signature (1,1), it will be seen in the next section that the union of the contact lifts $F \cup G$ is exactly the intersection of $I^\perp$ with the light-cone where $I = \langle \ell, \mathrm{d}\ell \rangle$. Similarly given an immersion $\ell : \Sigma^2 \to \mathbb{P}(\mathcal{L}^{4,2})$ (into the *Lie quadric*) of *definite* signature we may define $F \cup G := I^\perp \cap \mathcal{L}^{4,2}$ and ask what the conformal structures on Σ induced by F, G are. As described in [13, 22], such an ℓ corresponds to a sphere congruence (2-parameter family of spheres) in S^3 and that F, G correspond to the families of spheres in first order contact with surfaces f, g that *envelope* ℓ, see figure 2-3. F, G are then the contact lifts of f, g. Moreover the conformal structures on Σ induced by F, G have null directions the principal curvature directions of f, g. Just as the induced structures coincide when a line congruence is W, one defines a Ribaucour sphere congruence exactly when the curvature lines on the enveloping surfaces line up: indeed Eisenhart [28] describes 'transformations R of nets Ω' as exactly the transform $f \to g$ of enveloping surfaces of a sphere congruence where the curvature lines coincide.[6] The restrictions on the signatures in both cases is so that $\mathbb{P}(I^\perp \cap \mathcal{L})$ is not just the point ℓ. If one complexifies I and $\mathcal{L}$ then similar results can be obtained in terms of complex conjugate focal surfaces and enveloping surfaces.

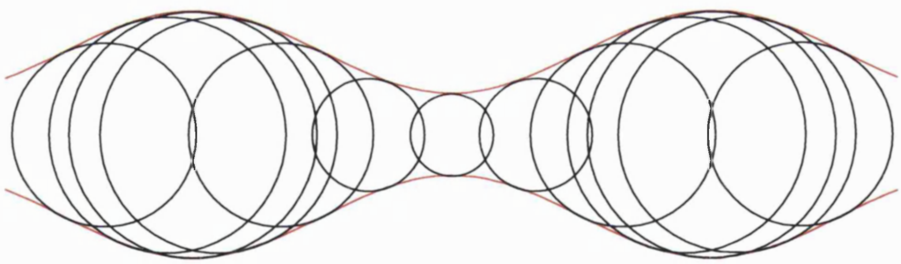

Figure 2-3: Sphere congruence with enveloping surfaces

2.6 The Conformal Gauss Map and the Normal Bundle

The discussion now becomes more modern as we consider the conformal geometry of line congruences. In conformal geometry there are a number of natural invariants of an immersion into a light cone, in particular we will concentrate on the Willmore

[6] The Ω *nets* are the enveloping surfaces f, g together with the (orthogonal) curvature line co-ordinates x, y.

energy and the curvature of the normal bundle. The main ingredient in this discussion is the *conformal Gauss map* [10] or *central sphere congruence*.[7]

Definition 2.12
The conformal Gauss map *of a non-degenerate line congruence* $\ell : \Sigma \to \mathbb{P}(\mathcal{L}^{3,3})$ *is the bundle of (2,2)-planes*

$$S = \langle \ell, \mathrm{d}\ell, \ell_{xy} \rangle$$

where x, y *are any conformal co-ordinates.*

It is clear that S is a conformal invariant of ℓ in that different choices of lift and of conformal co-ordinates make no difference to S. The synonymous expression *central sphere congruence* comes from the fact that the projective intersection of S with the light-cone $\mathcal{L}^{3,3}$ is a bundle of '(1,1)-spheres' $S^1 \times S^1/\mathbb{Z}_2$ [1] and that under any stereo projection,[8] the mean curvature vectors of S and ℓ coincide [11]. S is yet more fundamental than the definition suggests, for, as the following discussion shows, it can be naturally defined for any non-degenerate immersion into a light-cone without appealing to the existence of conformal co-ordinates.

Let $\sigma : \Sigma \to \mathcal{L}^{p,q}$ be a non-degenerate isometric immersion. The pull-back connection $\sigma^*\nabla$ on $T\Sigma \cong \sigma^* T\mathbb{R}^{p,q}$ induced by flat differentiation d is easily seen to be torsion-free and flat. The normal bundle $N\sigma := (\operatorname{Im} \mathrm{d}\sigma)^\perp$ is well-defined and complements $\operatorname{Im} \mathrm{d}\sigma$ by non-degeneracy. We can therefore split the pull-back connection into tangent and normal parts respectively:

$$(\sigma^*\nabla)_X(\mathrm{d}\sigma(Y)) = \mathrm{d}\sigma(\mathcal{D}_X Y) + \mathrm{I\!I}(X, Y).$$

It is easy to see that the connection $\mathcal{D}$ so defined is the Levi-Civita connection for $(\Sigma, (\mathrm{d}\sigma, \mathrm{d}\sigma))$. The normal part of the splitting is the *second fundamental form* $\mathrm{I\!I} \in \Gamma(S^2 T^*\Sigma \otimes N\sigma)$ of the immersion σ. The *mean curvature vector* is built out of the trace of $\mathrm{I\!I}$ with respect to $(\mathrm{d}\sigma, \mathrm{d}\sigma)$

$$H_\sigma := \frac{1}{\dim \Sigma} \operatorname{tr} \mathrm{I\!I},$$

while the *central sphere congruence* of σ is

$$S := \langle \sigma, \mathrm{d}\sigma, H_\sigma \rangle .$$

[7]Blaschke's *Zentralkugel* [6].

[8]The concept of stereo-projection in a generic light-cone, or indeed any symmetric R-space, is described in Chapter 5.

If $\{e_i\}$ is a basis of $T\Sigma$ with dual basis[9] $\{e^i\}$ with respect to σ then

$$(\sigma, H_\sigma) = \frac{1}{\dim\Sigma}\sum_{i=1}^{\dim\Sigma}(\sigma, \sigma^*\nabla_{e_i}(\mathrm{d}_{e^i}\sigma)) = -\frac{1}{\dim\Sigma}\sum_{i=1}^{\dim\Sigma}(\mathrm{d}_{e_i}\sigma, \mathrm{d}_{e^i}\sigma) = -1.$$

It follows that $\langle\sigma, H_\sigma\rangle$ is a bundle of (1,1)-planes perpendicular to $\operatorname{Im}\mathrm{d}\sigma$. Suppose that $\operatorname{Im}\mathrm{d}\sigma$ has signature (a,b): S is therefore a bundle of $(a+1, b+1)$-planes and so $\mathbb{P}(S\cap\mathcal{L}^{p,q}) \cong S^a \times S^b/\mathbb{Z}_2$ making S a congruence of (a,b)-spheres. Observe that under scalings of σ the mean curvature vector picks up tangent valued terms only and so S is invariant under scalings. It follows that S is well-defined and a conformal invariant for maps into the projective light-cone.

Specialising to the line congruence situation we see, once we have fixed conformal co-ordinates x, y and a choice of lift of ℓ, that the mean curvature vector is

$$H_\ell = \frac{1}{(\ell_x, \ell_y)}\mathrm{Proj}_{N\ell}\ell_{xy}$$

and so the central sphere congruence is $S = \left\langle \ell, \mathrm{d}\ell, \ell_{xy}\right\rangle$ as advertised.

Instead of considering lifts of ℓ we return to maps into the Klein quadric $\mathbb{P}(\mathcal{L}^{3,3})$ and decompose the tangent bundle of the quadric along ℓ. Thus

$$\ell^*T\mathbb{P}(\mathcal{L}^{3,3}) = \operatorname{Im}\mathrm{d}\ell \oplus N_\ell$$

where the rank 2 bundle N_ℓ is the *normal bundle*[10] of ℓ. As observed in (2.2) we have

$$\ell^*T_\ell\mathbb{P}(\mathcal{L}^{3,3}) \cong \hom(\ell, \ell^\perp/\ell).$$

Let $I = \langle\ell, \mathrm{d}\ell\rangle$ (independent of lift) so that $\operatorname{Im}\mathrm{d}\ell \cong \hom(\ell, I/\ell)$. It follows that $N_\ell \cong \hom(\ell, I^\perp/\ell)$. Define the *weightless normal bundle* by

$$\mathcal{N}_\ell = I^\perp/\ell$$

which differs from N_ℓ only up to tensoring by a line bundle. Since the Plücker images of the Laplace transforms $\ell^{\pm1}$ are perpendicular to ℓ and its derivatives (e.g. $(\ell^1, \ell_x) = \mathrm{vol}(\mathrm{g}\wedge\mathrm{g}^1\wedge(\mathrm{f}_x - b\mathrm{f})\wedge\mathrm{g}) = 0$) we see that

$$I^\perp = \left\langle \ell, \ell^{-1}, \ell^1\right\rangle.$$

[9] $(\mathrm{d}\sigma(e_i), \mathrm{d}\sigma(e^j)) = \delta_{ij}$.

[10] Not to be confused with $N\ell$ above: the rank 4 bundle normal to a specific *lift* of ℓ into the light-cone.

By calculation it is easy to see that $\langle \ell^{-1}, \ell^{1} \rangle = S^{\perp}$ where S is the conformal Gauss map defined above: we have therefore proved the following theorem.

Theorem 2.13
The Laplace transforms of a non-degenerate line congruence are the null directions in the weightless normal bundle to ℓ when canonically identified with $S^{\perp}$.

This theorem may be used as a definition of Laplace transform in any light cone in $\mathbb{R}^6$. Indeed given a non-degenerate immersion ℓ of a 2-manifold Σ into $\mathbb{P}(\mathcal{L}^{n,6-n})$ we may define $I = \langle \ell, \mathrm{d}\ell \rangle$ and the conformal Gauss map S and, since $I \subset S$, it is clear $S^{\perp} \subset I^{\perp}$. $S^{\perp}$ is therefore canonically identified with the weightless normal bundle by restriction of the projection $I^{\perp} \to I^{\perp}/\ell$ to $S^{\perp}$. The null directions of $S^{\perp}$ may then be taken as the definitions of two (possibly complex conjugate) Laplace transforms. In Lie sphere geometry a non-degenerate congruence of 2-spheres in S^3 corresponds to an immersion $\ell : \Sigma \to \mathbb{P}(\mathcal{L}^{4,2})$ where $\operatorname{Im} \mathrm{d}\ell$ has signature (2,0) or (1,1) and so Laplace transforms are either real or complex conjugate respectively. Indeed recalling our short discussion of sphere congruences with regard to contact lifts in Section 2.5 we see that real Laplace transforms of a sphere congruence ℓ share an enveloping surface with ℓ. In the case of non-degenerate maps into $\mathbb{P}(\mathcal{L}^{5,1}) \cong S^4$ the normal bundle has definite signature and Laplace transforms are forced to be complex conjugates.

2.7 Normal Curvature and the Willmore Density

In this section we build two modern invariants out of the bundle decomposition $\Sigma \times \mathbb{R}^{3,3} = S \oplus S^{\perp}$. By decomposing flat differentiation d in the trivial bundle $\Sigma \times \mathbb{R}^{3,3}$ we get connections on S and $S^{\perp}$, the curvatures of which are manifestly invariants of the congruence ℓ. Similarly the Willmore density of ℓ is easy to define as the pull-back of an invariant metric on $\hom(S, S^{\perp})$.

The weightless normal bundle $\mathcal{N}_\ell$ has a natural connection inherited from flat differentiation on $\mathbb{R}^{3,3}$: let $[\sigma] = \{\sigma + \lambda\ell\}_{\lambda \in \mathbb{R}}$ be a local section of $\mathcal{N}_\ell$ and set

$$\nabla^{\perp}[\sigma] = [\mathrm{Proj}_{I^{\perp}} \mathrm{d}\sigma].$$

This is easily seen to be a connection on $\mathcal{N}_\ell$. Indeed it is clear from $S^{\perp} \subset I^{\perp}$ that $\nabla^{\perp}$ is the connection on $\mathcal{N}_\ell$ obtained by the restriction of flat differentiation to $S^{\perp}$ under the identification $S^{\perp} \cong \mathcal{N}_\ell$. We compute the curvature of $\nabla^{\perp}$.

Let $\nabla^T, \nabla^{\perp}$ be the restrictions of flat differentiation on $\Sigma \times \mathbb{R}^{3,3}$ to $S, S^{\perp}$ respectively and define 1-forms $A \in \Omega^1_\Sigma \otimes \hom(S, S^\perp)$, $A' \in \Omega^1_\Sigma \otimes \hom(S^\perp, S)$ such that

$$\mathrm{d} = \nabla^T + \nabla^\perp + A + A'.$$

Expanding for the curvature of d gives four vanishing terms

$$\begin{aligned} 0 = \mathrm{d}^2 = (R^{\nabla^T} + A' \wedge A) + (R^{\nabla^\perp} + A \wedge A') \\ + (A \circ \nabla^T + \nabla^\perp \circ A') + (\nabla^T \circ A + A' \circ \nabla^\perp) \end{aligned}$$

where we have grouped terms according to the decomposition $\mathbb{R}^{3,3} = S \oplus S^\perp$. Upon choosing a lift of the focal surface f, and therefore lifts of ℓ, etc., it is easy to calculate $R^{\nabla^\perp} = -A \wedge A'$. Let $\sigma = p\ell^{-1} + q\ell^1$ be a local section of $S^\perp$ and let $X = \frac{\partial}{\partial x}, Y = \frac{\partial}{\partial y}$. Clearly $A \wedge A'(X,X) = 0 = A \wedge A'(Y,Y)$, so it remains to calculate the cross term:

$$\begin{aligned} A'_X\sigma &= \mathrm{Proj}_S(p\ell^{-1}_x + q\ell^1_x) \\ &= \mathrm{Proj}_S(p\mathrm{f}_{xx} \wedge \mathrm{f} + q(\mathrm{g}_x \wedge \mathrm{g}_y + \mathrm{g} \wedge \mathrm{g}_{xy})) \\ &= (h\alpha' q - \beta p)\mathrm{f} \wedge \mathrm{g} + (hq - \delta p)\mathrm{f} \wedge \mathrm{g}_y, \end{aligned}$$

$$\begin{aligned} A_Y A'_X \sigma &= \mathrm{Proj}_{S^\perp}(A'_X)_y \\ &= (hq - \delta p)\mathrm{Proj}_{S^\perp}(\mathrm{f}_y \wedge \mathrm{g}_y + \mathrm{f} \wedge \mathrm{g}_{yy}) \\ &= (hq - \delta p)(-\gamma'\ell^{-1} + \ell^1). \end{aligned}$$

Similarly

$$A_X A'_Y \sigma = (p - \gamma' q)(h\ell^{-1} - \delta\ell^1).$$

It follows that

$$R^{\nabla^\perp}(X,Y)\sigma = p(\delta\gamma' - h)\ell^{-1} + q(h - \delta\gamma')\ell^1.$$

Hence with respect to the null basis $\{\ell^{-1}, \ell^1\}$ of $S^\perp$ we have

$$R^{\nabla^\perp} = W \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mathrm{d}x \wedge \mathrm{d}y \qquad (2.16)$$

where x is the co-ordinate associated to the $-ve$ entry in the matrix: i.e. to the Laplace transform $\ell^{-1} = \mathrm{f} \wedge \mathrm{f}_x$. It is therefore clear (and reassuring) that swapping the roles of the co-ordinates x, y has no effect on W if we choose to *define* W this way.

We can do better by observing a canonical duality between 2-forms and symmetric

2-tensors in the conformal class of $\mathrm{d}x\mathrm{d}y$. Recall the line subbundles L_-, L_+ of $T\Sigma$ defined in (2.5). Define the bundle endomorphism $J \in \operatorname{End} T\Sigma$ by $J = \mp\mathrm{Id}$ on L_-, L_+ respectively and observe that since $L_\pm$ are the null directions of $(\mathrm{d}\ell, \mathrm{d}\ell)$ we have

$$g(X, JY) = g(X_+ + X_-, Y_+ - Y_-) = -g(X_+ - X_-, Y_+ + Y_-) = -g(JX, Y)$$

and so J is skew for any representative metric g in the conformal class of $(\mathrm{d}\ell, \mathrm{d}\ell)$. It follows that

$$(X, Y) \mapsto R^{\nabla^\perp}(X, JY)$$

is the symmetric 2-tensor $\hat{J}\mathcal{W} \in \Gamma(S^2T^*\Sigma \otimes \operatorname{End} S^\perp)$ where $\hat{J} = \pm 1$ on $\ell^{\pm 1}$ respectively. This association is consistent with the labelling of the Laplace transforms as positive and negative from the outset. Indeed the labelling of the Laplace transforms *defines* which of the line subbundles of $T\Sigma$ are positive and negative for if $f := \ell \cap \ell^{-1}$ then we define L_- as the line subbundle such that $f \wedge \mathrm{d}f(L_-) = \ell^{-1}$. The Weingarten invariant may therefore be defined without reference to focal surfaces and, up to sign, without reference to which Laplace transform is positive.

We have proved that W-congruences correspond to surfaces in the Klein quadric with flat normal bundle. An avatar of this can be found in Eisenhart [28] and Ferapontov [29] where W-congruences are shown to have Plücker co-ordinates satisfying a Laplace equation (e.g. 2.8). Indeed it is easy to see that the co-ordinates on Σ for which one obtains the Laplace equations are asymptotic for the focal surfaces f, g. The asymptotic directions are, of course, coincident iff ℓ is a W-congruence.

One may perform the same calculations for immersions into the Lie quadric: the curvature of the normal bundle of a non-degenerate sphere congruence may not have a classical name but it is an invariant of the congruence. Moreover it is known (e.g. [28, 29]) that a sphere congruence has flat normal bundle iff the congruence is Ribaucour. Indeed Lie's line-sphere correspondence [28], taking line congruences to sphere congruences, is known to send W-congruences to Ribaucour congruences.

The conformal Gauss map of a non-degenerate congruence gives rise to a further invariant 2-form which will be seen to be conformal to $\mathcal{H}$ and $\mathcal{W}$. Define a metric on $\hom(S, S^\perp)$ by

$$(A, B) := \operatorname{tr}(A^*B)$$

where $*$ is the adjoint. Both $S, S^\perp$ are non-degenerate, so $(\ ,\ )$ is well-defined. Being a map into a non-degenerate Grassmannian, $S : \Sigma \to G_{(2,2)}(\mathbb{R}^{3,3})$ has tangent bundle $TS \cong \hom(S, S^{\perp})$ and so we can pull-back $(\ ,\ )$ via S to get a well-defined metric on Σ. $\mathrm{d}S$ is the map

$$T\Sigma \ni X \mapsto (\sigma \mapsto \mathrm{Proj}_{S^{\perp}} \mathrm{d}_X \sigma)$$

where σ is a local section of S. Specifically the tangent vector $\frac{\partial}{\partial x}$ gets mapped by $\mathrm{d}S$ to

$$S_x : \sigma \mapsto \mathrm{Proj}_{S^{\perp}} \left(\frac{\partial \sigma}{\partial x} \right).$$

Let $\sigma = p\ell + q\ell_x + r\ell_y + s\ell_{xy}$ be a local section of S given in terms of some lift $\ell = \mathrm{f} \wedge \mathrm{g}$. Then

$$S_x \sigma = \mathrm{Proj}_{S^{\perp}} (q\ell_{xx} + s\ell_{xxy}) \Rightarrow \ker S_x \supseteq \langle \ell, \ell_y \rangle$$

and so

$$\mathrm{Im}\, S_x^* \subseteq \mathrm{Proj}_S \langle \ell, \ell_y \rangle^{\perp} = \langle \ell, \ell_y \rangle,$$

since $\langle \ell, \ell_y \rangle$, being 2-dimensional, is maximally isotropic in S. The endomorphism $S_x^* \circ S_x$ is therefore identically zero: similarly for S_y. The induced metric on Σ thus depends only on the cross term. Now $\ker S_x \supseteq \langle \ell, \ell_y \rangle$ and $\mathrm{Im}\, S_y^* \subseteq \langle \ell, \ell_x \rangle$, so we need only evaluate on ℓ_x to calculate the trace of $S_y^* S_x$.

$$\begin{aligned}
(S_x, S_y) &= \mathrm{tr}\, S_y^* S_x = \mathrm{coeff}_{\ell_x}(S_y^* S_x \ell_x) \\
&= \mathrm{coeff}_{\ell_x}(S_y^* \mathrm{Proj}_{S^{\perp}} \ell_{xx}) = -\,\mathrm{coeff}_{\ell_x}(S_y^*(h\ell^{-1} + \delta\ell^{1})) \\
&= -\,\mathrm{coeff}_{\ell_x}(h(\mathrm{g} \wedge \mathrm{f}_x - b\mathrm{f} \wedge \mathrm{f}_y) + \delta\gamma' \mathrm{g} \wedge \mathrm{f}_x) \\
&= h + \delta\gamma'.
\end{aligned}$$

We therefore have a third invariant symmetric 2-tensor $C := (h + \delta\gamma')\, \mathrm{d}x \mathrm{d}y$ on Σ in the same conformal class as the Laplace and Weingarten invariants. Indeed since we have seen that $\mathcal{W}$ and C can be defined without reference to focal surfaces, the relation

$$\mathcal{H} = \frac{1}{2}(C + \mathcal{W})$$

may serve as a *definition* of the Laplace invariant of a line congruence.

We are now in a position to make a strong case for a conformal redefinition of the Laplace invariant of a line congruence as a genuine 2-form. The 2-form $\mathcal{E} := C(-, J-) = (h + \delta\gamma')\mathrm{d}x \wedge \mathrm{d}y = (h + \delta\gamma')\mathrm{d}A \in \Omega^2_{\Sigma}$ is exactly the *Willmore density*

$(H^2 - K)\,dA$ [13] of ℓ. With only the choice of which Laplace transform is positive we see that the curvature $R^{\nabla^\perp}$ of the normal bundle is the 2-form $\mathcal{R} := W dA$ and so we may define the Laplace invariant of ℓ to be the 2-form

$$\hat{\mathcal{H}} := \frac{1}{2}(\mathcal{E} + \mathcal{R}) = h\,dA.$$

The sign of $\hat{\mathcal{H}}$ obviously depends on the orientation of Σ. We also have the question of what is $\frac{1}{2}(\mathcal{E} - \mathcal{R})$ obtained by reversing the choice of which Laplace transform is positive. This amounts to finding a more geometric description of

$$\delta\gamma'\,dxdy = \frac{1}{2}(C - \mathcal{W}),$$

which will be tackled in the next section.

2.8 Dual Congruence

Definition 2.4 introduced the dual of a focal surface in $\mathbb{P}^3$ as the annihilator of its tangent bundle: $f^* := \mathrm{Ann}\,\langle f, df\rangle$. The *dual congruence* to $\ell = f \wedge g$ is the line congruence ℓ^* in $\mathbb{P}^3_*$ obtained by joining the two dual surfaces f^*, g^*. ℓ^* can be viewed more invariantly as the annihilator of ℓ since

$$\ell = \langle f, df\rangle \cap \langle g, dg\rangle = f \wedge g \Longrightarrow \ell^* = f^* \wedge g^* = \mathrm{Ann}\,f \cap \mathrm{Ann}\,g.$$

More is true, for differentiating the relations $\mathsf{f}^*\mathsf{f} = \mathsf{f}^*\mathsf{g} = \mathsf{g}^*\mathsf{f} = \mathsf{g}^*\mathsf{g} = 0$ gives

$$\mathsf{f}^*_x\mathsf{f} = \mathsf{f}^*_x\mathsf{g} = \mathsf{g}^*_y\mathsf{f} = \mathsf{g}^*_y\mathsf{g} = 0 \Longrightarrow \mathsf{f}^*_x, \mathsf{g}^*_y \in \ell^*.$$

f^*, g^* are therefore the focal surfaces of the dual congruence for which x, y are conformal co-ordinates. Notice that the roles of x, y have swapped over

$$\mathsf{f}_y, \mathsf{g}_x \in \ell \leftrightsquigarrow \mathsf{f}^*_x, \mathsf{g}^*_y \in \ell^*.$$

Fix lifts of the dual surfaces by forcing $\mathsf{f}^*\mathsf{g}^1 = 1 = \mathsf{g}^*\mathsf{f}^{-1}$. Indeed by calculating ($\mathsf{f}^*_x\mathsf{v} = (\mathsf{f}^*\mathsf{v})_x - \mathsf{f}^*\mathsf{v}_x$, etc.) it is easy to see that we have the equivalent of (2.7)

$$\mathsf{f}^*_x = b\mathsf{f}^* - \delta\mathsf{g}^*, \qquad \mathsf{g}^*_y = -\gamma'\mathsf{f}^* + a\mathsf{g}^*, \tag{2.17}$$

and so

$$
\begin{aligned}
&\mathrm{f}^*_{xy} + a^*\mathrm{f}^*_x + b^*\mathrm{f}^*_y + c^*\mathrm{f}^* = 0, \\
&\qquad a^* = -a - (\ln\delta)_y, \qquad b^* = -b, \\
&\qquad c^* = b(a + (\ln\delta)_y) - b_y - \delta\gamma', \\
&\mathrm{g}^*_{xy} + a^*_1\mathrm{g}^*_x + b^*_1\mathrm{g}^*_y + c^*_1\mathrm{g}^* = 0, \\
&\qquad a^*_1 = -a, \qquad b^*_1 = -b - (\ln\gamma')_x, \\
&\qquad c^*_1 = a(b + (\ln\gamma')_y) - a_x - \delta\gamma', \\
&h^* = a^*b^* - c^* + b^*_y = \delta\gamma', \\
&k^* = a^*b^* - c^* + a^*_x = \delta\gamma' - b_y + \gamma_y + (a_1)_x - \delta'_x + (\ln h/\delta)_{xy}.
\end{aligned}
\tag{2.18}
$$

Since the roles of x, y have swapped over we must add b^*_y, not a^*_x as previously, to calculate h^*. The geometric meaning of the mystery $\delta\gamma'\mathrm{d}x\mathrm{d}y$ is now clear: it is the Laplace invariant $\mathcal{H}^*$ of the dual congruence ℓ^*. Furthermore $\mathcal{W} = \mathcal{H} - \mathcal{H}^*$ and so the Weingarten invariant $\mathcal{W}^*$ of ℓ^* is simply $-\mathcal{W}$, while the metric on Σ induced by the central sphere congruence is $C^* = C$. $\mathcal{W}^* = -\mathcal{W}$ can be seen directly from our expression (2.16) for the curvature of the normal bundle: the entire calculation of $R^{\nabla^\perp_*}$ is the same as that of $R^{\nabla^\perp}$ except for the fact that the role of the co-ordinates has reversed and so the only change is $\mathrm{d}x \wedge \mathrm{d}y \mapsto \mathrm{d}y \wedge \mathrm{d}x$.

It is a matter of calculation to see that Laplace transforms commute with duality: i.e. $(\mathrm{f}^*)^{-1} := \mathrm{f}^*_y + a^*\mathrm{f}^*$ is the dual surface to f^{-1} and $(\mathrm{g}^*)^1 := \mathrm{g}^*_x + b^*_1\mathrm{g}^*$ is dual to g^1. The following diagram therefore commutes:

$$
\begin{tikzcd}
{} & f^{-1*} \arrow[l, "y"'] \arrow[r, "x"'] \arrow[dd, leftrightarrow] & \ell^{-1*} & f^* \arrow[l, "y"'] \arrow[r, "x"'] \arrow[dd, leftrightarrow] & \ell^* & g^* \arrow[l, "y"'] \arrow[r, "x"'] \arrow[dd, leftrightarrow] & \ell^{1*} & g^{1*} \arrow[l, "y"'] \arrow[r, "x"'] \arrow[dd, leftrightarrow] & {} \\
\cdots & & & & & & & & \cdots \\
{} & f^{-1} \arrow[l, "x"'] \arrow[r, "y"'] & \ell^{-1} & f \arrow[l, "x"'] \arrow[r, "y"'] & \ell & g \arrow[l, "x"'] \arrow[r, "y"'] & \ell^{1} & g^{1} \arrow[l, "x"'] \arrow[r, "y"'] & {}
\end{tikzcd}
$$

Similarly to equations (2.13,2.14) we have the following recurrence relations for the dual Laplace invariants h^*_i and coefficients a^*_i, b^*_i, c^*_i

$$
\begin{aligned}
&a^*_{i+1} = a^*_i, \qquad b^*_{i+1} = b^*_i - (\ln h^*_i)_x, && \multirow{2}{*}{$i \geq 0,$} \\
&c^*_{i+1} = c^*_i + (a^*_i)_x - (b^*_i)_y - a^*_i(\ln h^*_i)_x, && i \geq 0, \\
&a^*_{i-1} = a^*_i - (\ln k^*_{|i|})_y, \qquad b^*_{i-1} = b^*_i, && \\
&c^*_{i-1} = c^*_i - (a^*_i)_x + (b^*_i)_y - b^*_i(\ln k^*_{|i|})_y, && i \leq 0, \\
&h^*_{i+1} + h^*_{i-1} = 2h^*_i - (\ln h^*_i)_{xy}, && \forall i, \\
&h^*_{i+1} = h^*_i + h^*_1 - h^*_0 - (\ln(h^*_1 \cdots h^*_i))_{xy}, && \\
&k^*_{i+1} = k^*_i + k^*_1 - k^*_0 - (\ln(k^*_1 \cdots k^*_i))_{xy}. && i > 0,
\end{aligned}
$$

while the Weingarten invariants and metrics C_i satisfy

$$\begin{aligned}
C_{i+1} + C_{i-1} &= 2C_i - (\ln h_i h_i^*)_{xy}\, dx dy, \\
C_{i+1} &= C_i + C_1 - C - (\ln h_1 \cdots h_i h_1^* \cdots h_i^*)_{xy}\, dx dy, \\
\mathcal{W}_{i+1} + \mathcal{W}_{i-1} &= 2\mathcal{W}_i - (\ln h_i / h_i^*)_{xy}\, dx dy, \\
\mathcal{W}_{i+1} &= \mathcal{W}_i + \mathcal{W}_1 - \mathcal{W} - \left(\ln \frac{h_1 \cdots h_i}{h_1^* \cdots h_i^*}\right)_{xy} dx dy.
\end{aligned}$$

As a corollary to the second last equation we obtain a classical result: if two consecutive $\mathcal{W}_i$ are zero, then all congruences in the Laplace sequence are W (since $\mathcal{W}_i = 0 \Rightarrow h_i = h_i^*$).

2.9 The Structure Equations

Suppose ℓ is non-degenerate and fix a lift of f so that $\mathbf{e} = (\mathrm{f}, \mathrm{g}, \mathrm{f}^{-1}, \mathrm{g}^1)^T$ is a moving frame of $\mathbb{R}^4$. Define a 1-form $\omega \in \Omega^1_\Sigma \otimes \mathfrak{gl}(4)$ by $d\mathbf{e} = \omega \mathbf{e}$. Because of (2.12) and the easily verifiable

$$\mathrm{f}^{-1}_y = k\mathrm{f} - a\mathrm{f}^{-1}, \qquad \mathrm{g}^1_x = h^1 \mathrm{g} - b\mathrm{g}^1,$$

it is seen that

$$\omega = \begin{pmatrix} -b & 0 & 1 & 0 \\ h & -b & 0 & 0 \\ \alpha & \beta & \gamma & \delta \\ 0 & h^1 & 0 & -b \end{pmatrix} dx + \begin{pmatrix} -a & 1 & 0 & 0 \\ 0 & -a^1 & 0 & 1 \\ k & 0 & -a & 0 \\ \alpha' & \beta' & \gamma' & \delta' \end{pmatrix} dy =: A dx + B dy.$$

The structure equations are $0 = d^2\mathbf{e} = d\omega \mathbf{e} - \omega \wedge d\mathbf{e}$: i.e. $d\omega = \omega \wedge \omega$ or equivalently $-A_y + B_x = [A, B]$. Now

$$[A, B] = \begin{pmatrix} k - h & 0 & 0 & 0 \\ h(a^1 - a) & h - h^1 & 0 & 0 \\ k(b+\gamma) + \delta\alpha' & \alpha + \delta\beta' + \beta(a - a^1) & \delta\gamma' - k & \beta + \delta(a + \delta') \\ -\alpha\gamma' - h\beta' & -h^1(a^1 + \delta') - \beta\gamma' & -\alpha' - \gamma'(b+\gamma) & h^1 - \delta\gamma' \end{pmatrix},$$

and so the structure equations are therefore

1. $b_y - a_x = k - h,$
2. $h_y = h(a - a^1),$

3. $b_y - a^1_x = h - h^1$,
4. $\alpha_y = k_x - k(b+\gamma) - \delta\alpha'$,
5. $\beta_y = -\alpha - \delta\beta' + \beta(a^1 - a)$,
6. $\gamma_y = k - \delta\gamma' - a_x \quad (= W + b_y - 2a_x)$,
7. $\delta_y = -\beta - \delta(a + \delta')$,
8. $\alpha'_x = -\alpha\gamma' - h\beta'$,
9. $\beta'_x = h^1_y - h^1(a^1 + \delta') - \beta\gamma'$,
10. $\gamma'_x = -\alpha' - \gamma'(b+\gamma)$,
11. $\delta'_x = h^1 - \delta\gamma' - b_y \quad (= W + a^1_x - 2b_y)$.

The first three equations tell us nothing new, although a number of the remainder will be useful for the applications below. For our purposes it is not really necessary to calculate all the structure equations in the above manner since they are all readily available as 'mixed partial derivatives commute' conditions, but for completeness we record them all. Since the structure equations are all that are required to define a line congruence, many people have approached this subject by simply defining a line congruence as a solution to the structure equations. A good example of this is the Wilczynski-frame [64] where the author chooses a specific lift of the focal surfaces for which an even simpler set of structure equations than ours is available.

Isothermic congruences

Definition 2.14
Conjugate co-ordinates x, y on a surface f are strictly isothermic conjugate *if $x \pm y$ are asymptotic co-ordinates (i.e. null for $\mathbb{I}$).*

Remarks: In our notation strictly isothermic means $\delta = -1$. We will refer to co-ordinates x, y as isothermic conjugate if there exist monotone functions u, v such that $u(x), v(y)$ are strictly isothermic conjugate.

Theorem 2.15 (Demoulin–Tzitzeica [25, 58])
Let f be a focal surface of a W-congruence $\ell = f \wedge f_y$ with conformal co-ordinates x, y. The Laplace transform $\ell^{-1} = f \wedge f_x$ is a W-congruence iff x, y are isothermic conjugate on f.

Proof Calculating W_{-1} via our expression (2.18) for k^* and the 6th and 11th structure equations yields

$$W_{-1} = W + (\ln\delta)_{xy}.$$

The theorem now reads 'isothermic conjugate iff $(\ln\delta)_{xy} = 0$'. Since any logarithm of -1 is constant it is clear that strictly isothermic co-ordinates satisfy $(\ln\delta)_{xy} = 0$.

Conversely, suppose that $\delta < 0$ and $(\ln\delta)_{xy} = 0$, then we can choose a logarithm of δ such that $\ln\delta = U(x) + V(y) + i\pi$ where U, V are real functions. Let $U(x) = 2\ln u'(x)$, $V(y) = -2\ln v'(y)$ for some functions u, v with positive derivatives. Then

$$\delta = -\frac{u'(x)^2}{v'(y)^2}.$$

It is easy to see that in the new co-ordinates u, v we have $\hat{\mathrm{g}} = \frac{\mathrm{d}y}{\mathrm{d}v}\mathrm{g}$, $\hat{\mathrm{f}}^{-1} = \frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{f}^{-1}$, $\hat{\mathrm{g}}^1 = \left(\frac{\mathrm{d}y}{\mathrm{d}v}\right)^2\mathrm{g}^1$, $\hat{\mathrm{f}}^* = \left(\frac{\mathrm{d}v}{\mathrm{d}y}\right)^2\mathrm{f}^*$, etc. and so

$$\hat{\mathrm{f}}^*\hat{\mathrm{f}}_{uu} = \hat{\mathrm{f}}^*\hat{\mathrm{f}}_u^{-1} = \delta\left(\frac{v'}{u'}\right)^2\mathrm{f}^*\mathrm{g}^1 = -1, \qquad \hat{\mathrm{f}}^*\hat{\mathrm{f}}_{vv} = \mathrm{f}^*\mathrm{g}^1 = 1.$$

u, v are therefore isothermic conjugate as in the definition.

If $\delta > 0$ and $(\ln\delta)_{xy} = 0$ then $\mathrm{I\!I}_f$ is definite and so asymptotic co-ordinates are complex. By a similar process as above we can find conjugate u, v such that $\delta = 1$, the asymptotic co-ordinates are then $u \pm iv$ and so u, iv are isothermic.

■

With half an eye on chapters 5 and 6 we make the following definition.

Definition/Theorem 2.16
A line congruence ℓ is isothermic *if there exists conformal co-ordinates x, y which are strictly isothermic conjugate on both focal surfaces. Equivalently ℓ is a W-congruence with x, y isothermic on both surfaces.*

The implied theorem is that this definition of isothermic is consistent with that to be given in Chapter 5 and with any other of which the reader may be aware. We shall prove this in Section 6.2.

By the Demoulin–Tzitzeica theorem we see that the Laplace transforms $\ell^{\pm 1}$ of an isothermic congruence are W-congruences and that consequently *all* Laplace transforms are W. Furthermore the same theorem says that x, y are isothermic conjugate on every focal surface. The entire Laplace sequence of an isothermic congruence is therefore isothermic.

None of this is new: Tzitzeica [58] defined an R *net* to be a surface f with conjugate co-ordinates x, y such that $f \wedge f_x$ and $f \wedge f_y$ are both W-congruences. In such a case either of the W-congruences tangent to f are named *R-congruences.* The discussion in section 45 of Eisenhart [28] shows that an R net has x, y isothermic conjugate (Tzitzeica) and conversely that if $f \wedge f_x$ is W where x, y are isothermic conjugate on f

then (f,x,y) is an R net (Demoulin). Thus R-congruence is a synonym for isothermic. Eisenhart also demonstrates that the Laplace transforms of an R net are R nets. All we have done is to obtain the same results in a different manner.

Example: Our earlier hyperboloid example (figure 2-1) has $\delta = 1$ and so the coordinates x, iy are isothermic on f. Since all congruences in the Laplace sequence are W it follows that all four congruences are isothermic.

Line congruences in a linear complex

Definition 2.17

A linear complex [28] is a 3-parameter family of lines in $\mathbb{P}^3$ whose Plücker images lie in a hyperplane in $\mathbb{P}^5$. A line congruence ℓ is said to lie in a linear complex iff the Plücker images of ℓ lie on a hyperplane.

Suppose ℓ lies in a linear complex, then so do its derivatives. The central sphere congruence S is therefore in the supplied hyperplane which must therefore be perpendicular to a section of the normal bundle $S^\perp$. Let

$$\xi = p\mathrm{f} \wedge \mathrm{f}^{-1} + q\mathrm{g} \wedge \mathrm{g}^1$$

be such a section of $S^\perp = \langle \ell^{-1}, \ell^1 \rangle$. If the hyperplane is to be constant we must have

$$\xi_x \equiv 0 \equiv \xi_y \pmod{\xi}.$$

Calculate

$$\begin{aligned}
\xi_x &= p\beta \mathrm{f} \wedge \mathrm{g} + (p\delta + qh)\mathrm{f} \wedge \mathrm{g}^1 + (p_x + p(\gamma - b))\mathrm{f} \wedge \mathrm{f}^{-1} + (q_x - 2qb)\mathrm{g} \wedge \mathrm{g}^1, \\
\xi_y &= q\alpha' \mathrm{f} \wedge \mathrm{g} + (p + q\gamma')\mathrm{g} \wedge \mathrm{f}^{-1} + (p_y - 2pa)\mathrm{f} \wedge \mathrm{f}^{-1} + (q_y + q(\delta' - a^1))\mathrm{g} \wedge \mathrm{g}^1.
\end{aligned}$$

It is clear that both p, q are non-zero, for otherwise $h = 0$ and ℓ is degenerate. We therefore require

$$\begin{aligned}
&-\frac{p}{q} = \frac{h}{\delta} = \gamma', \qquad \beta = 0 = \alpha', \\
&qp_x - pq_x + (\gamma + b)pq = 0 = pq_y - qp_y - (\delta' + 2a - a^1)pq.
\end{aligned}$$

The second line follows from the first by differentiating and applying the 7th and 10th structure equations. In conclusion we have obtained a theorem of Sasaki [51] in simplified form:

Theorem 2.18
A line congruence ℓ lies in a linear complex iff $W = \beta = \alpha' = 0$.

The following result is also implied, but not proved, by Sasaki.

Corollary 2.19
A line congruence belongs to a linear complex iff ℓ and ℓ^* are projectively equivalent: i.e. there exists a constant linear map $T : \mathbb{R}^4 \to \mathbb{R}^4_*$ such that $T\ell = \ell^*$.

Proof From the 6^{th} and 11^{th} structure equations $((\gamma - b)_y = -2a_x, (\delta' - a^1)_x = -2b_y)$ we can choose functions λ, μ such that

$$(\ln \lambda)_x = -2b, \quad (\ln \lambda)_y = \delta' - a^1, \quad (\ln \mu)_x = \gamma - b, \quad (\ln \mu)_y = -2a. \tag{2.19}$$

It follows that

$$\operatorname{grad}\left(\ln \frac{\lambda}{\mu}\right) = \operatorname{grad}\left(\ln \frac{p}{q}\right) \Rightarrow \frac{\lambda}{\mu} = C\frac{p}{q},$$

for some constant C. We have freedom up to scaling by constants in λ, μ, so we can choose λ, μ such that $C = 1$.

Fix a lift of f and consider the lifts of the dual surfaces f^*, g^* given by

$$\hat{\mathrm{f}}^* = \lambda \mathrm{f}^*, \qquad \hat{\mathrm{g}}^* = \mu \mathrm{g}^*.$$

It is easy to see that

$$\begin{aligned} \hat{\mathrm{f}}^*_x &= ((\ln \lambda)_x + b)\hat{\mathrm{f}}^* - \delta\frac{\lambda}{\mu}\hat{\mathrm{g}}^* = h\hat{\mathrm{g}}^* - b\hat{\mathrm{f}}^*, \\ \hat{\mathrm{g}}^*_y &= \hat{\mathrm{f}}^* - a\hat{\mathrm{g}}^*. \end{aligned}$$

Comparing this with $\mathrm{f}_y = \mathrm{g} - a\mathrm{f}$, $\mathrm{g}_x = h\mathrm{f} - b\mathrm{g}$ we see that f and $\hat{\mathrm{g}}^*$ have the same conjugate net equation, as do $\mathrm{g}, \hat{\mathrm{f}}^*$. We can also calculate Laplace transforms $(\hat{\mathrm{f}}^*)^{-1} := \hat{\mathrm{f}}^*_y + a^1\hat{\mathrm{f}}^*$, $(\hat{\mathrm{g}}^*)^1 := \hat{\mathrm{g}}^*_x + b\hat{\mathrm{g}}^*$ and their affect on the moving frame $(\mathrm{f}, \mathrm{g}, \mathrm{f}^{-1}, \mathrm{g}^1)$. Thus

$$(\hat{\mathrm{f}}^*)^{-1}\begin{pmatrix} \mathrm{f} \\ \mathrm{g} \\ \mathrm{f}^{-1} \\ \mathrm{g}^1 \end{pmatrix} = \begin{pmatrix} 0 \\ -\lambda \\ 0 \\ \lambda_y + \lambda(a^1 - \delta') \end{pmatrix} = \begin{pmatrix} 0 \\ -\lambda \\ 0 \\ 0 \end{pmatrix}, \qquad (\hat{\mathrm{g}}^*)^1\begin{pmatrix} \mathrm{f} \\ \mathrm{g} \\ \mathrm{f}^{-1} \\ \mathrm{g}^1 \end{pmatrix} = \begin{pmatrix} -\mu \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

It is now simple to see that

$$(\hat{\mathrm{f}}^*)^{-1}_y = \beta'\hat{\mathrm{g}}^* + \gamma'(\hat{\mathrm{f}}^*)^{-1} + \delta'(\hat{\mathrm{g}}^*)^1,$$

$$(\hat{g}^*)^1_x = \alpha\hat{f}^* + \gamma(\hat{f}^*)^{-1} + \delta(\hat{g}^*)^1,$$

which are exactly the same equations satisfied by g^1_y, f^{-1}_x respectively. Define a linear map $T(x,y) : \mathbb{R}^4 \to \mathbb{R}^{4*}$ by $T : (f, g, f^{-1}, g^1) \mapsto (\hat{g}^*, \hat{f}^*, (\hat{g}^*)^1, (\hat{f}^*)^{-1})$. The above equations are exactly what are required to guarantee that T is constant. Hence there exists a constant projectivity mapping ℓ to its dual and so ℓ, ℓ^* are projectively equivalent.

For the converse note that focal surfaces and conjugate co-ordinates are preserved by a constant T and so $T(f, g, f^{-1}, g^1) = (g^*, f^*, (g^*)^1, (f^*)^{-1})$. Define $\lambda := (Tg)g^1$, $\mu := (Tf)f^{-1}$ and evaluate $(Tg)_x = h(Tf) - b(Tg)$ and $(Tf)_y = Tg - a(Tf)$ on f^{-1}, g^1 respectively to obtain

$$\lambda = -\gamma'\mu, \qquad \lambda\delta = -h\mu, \qquad (\ln\lambda)_x = -2b, \qquad (\ln\mu)_y = -2a.$$

Thus ℓ is a W-congruence and we can cross-differentiate using the structure equations to see that

$$(\ln\lambda)_y = \delta' - a^1 + \frac{\beta}{\delta}, \qquad (\ln\mu)_x = \gamma - b + \frac{\alpha'}{\gamma'}.$$

Now evaluate $(Tf)^1_x = \alpha(Tf) + \beta(Tg) + \gamma(Tf)^{-1} + \delta(Tg)^1$ on f to see that $\mu_x = \mu(\gamma - b) \Rightarrow \alpha' = 0$. Similarly $\beta = 0$ and so λ, μ satisfy (2.19) and the theorem is proved.

■

2.10 The Euler–Darboux Equation

We conclude this chapter with another example, again inspired by Sasaki. The Euler–Darboux equation $E(m,n)$ can be expressed as follows

$$f_{xy} + \frac{n}{x-y}f_x - \frac{m}{x-y}f_y = 0, \tag{2.20}$$

where m, n are constant. A solution can be found by separation of variables, yielding the following family of conjugate nets in $\mathbb{P}^3$ satisfying the Euler–Darboux equation. Let a, b, c, d be distinct constants, then

$$f(x,y) = [(x-a)^m(y-a)^n, (x-b)^m(y-b)^n, (x-c)^m(y-c)^n, (x-d)^m(y-d)^n]$$

are the homogeneous co-ordinates of a surface satisfying (2.20): the surfaces f constitute exactly those separable solutions of $E(m,n)$.

It is easy[11] to see that the Laplace transforms of f (where they make sense) have the following homogeneous representations

$$\begin{aligned} f^i(x,y) =&[(x-a)^{m+i}(y-a)^{n-i},(x-b)^{m+i}(y-b)^{n-i},\\ &\qquad (x-c)^{m+i}(y-c)^{n-i},(x-d)^{m+i}(y-d)^{n-i}]. \end{aligned}$$

Notice that f^i is a solution to $E(m+i,n-i)$. By calculating determinants ($\mathrm{f}^* \propto \mathrm{f}\wedge \mathrm{f}_x \wedge \mathrm{f}_y$) the dual surface can also be expressed in homogeneous co-ordinates:

$$\begin{aligned} f^* =&[C_{abc}(x-d)^{1-m}(y-d)^{1-n},C_{abd}(x-c)^{1-m}(y-c)^{1-n},\\ &\qquad C_{acd}(x-b)^{1-m}(y-b)^{1-n},C_{bcd}(x-a)^{1-m}(y-a)^{1-n}] \end{aligned}$$

where C_{pqr} is the cyclic expression $(p-q)(q-r)(r-p)$ for any p,q,r. By weighting co-ordinates on $\mathbb{P}^3_*$ we can fix all constants $C=1$: the dual surface is therefore a solution to $E(1-m,1-n)$. The Laplace invariants of the line congruences $\ell^i := f^i \wedge f^{i+1}$ are calculable from the Euler–Darboux equations themselves:

$$\mathcal{H}^i = -\frac{(m+i+1)(n-i)}{(x-y)^2}\,\mathrm{d}x\mathrm{d}y, \qquad \mathcal{H}^{*i} = -\frac{(m-1+i)(n-2-i)}{(x-y)^2}\,\mathrm{d}x\mathrm{d}y,$$

and so

$$\mathcal{W}^i = \mathcal{H}^i - \mathcal{H}^{*i} = 2\frac{1-m-n}{(x-y)^2}\,\mathrm{d}x\mathrm{d}y,$$

which is independent of i. Thus *any* ℓ^i is a W-congruence $\iff$ *all* are $\iff$ $m+n=1$.

This example is easy to work with since we only need to change the indices in our expressions to find all relevant surfaces:

$$\begin{tikzcd} f:(m,n) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & f^i:(m+i,n-i) \arrow[d, leftrightarrow] \\ f^*:(1-m,1-n) \arrow[r, leftrightarrow] & f^{*i}:(1-m-i,1-n+i) \end{tikzcd}$$

By similar determinant calculations to the above we see that

$$\delta = -\frac{m(m-1)(y-a)(y-b)(y-c)(y-d)}{n(n-1)(x-a)(x-b)(x-c)(x-d)} \implies (\ln\delta)_{xy} = 0.$$

x,y are therefore isothermic conjugate on all focal surfaces, although not strictly so.

[11] For Maple buffs!

Indeed by Demoulin–Tzitzeica (Theorem 2.15) ℓ is an isothermic congruence iff $W = 0$ iff all Laplace transforms are isothermic.

Similarly it can be seen that $\alpha' = \beta = 0 \iff$ two of a, b, c, d are equal in which case the congruence lives in a $\mathbb{P}^2$. This links up with the Klein correspondence, where the set of lines lying in a hyperplane in $\mathbb{P}^3$ corresponds to a null (projective) 2-plane in the Klein quadric: a so-called β-plane. We therefore see that ℓ lies in a linear complex iff it actually lies in a fixed β-plane.

2.11 Further Work

There are a plethora of classical questions that can be asked of line congruences to which answers are known: i.e. investigations of Laplace sequences that terminate, special types (e.g. quadratic) of focal surfaces, consideration of affine minimal surfaces via W-congruences. Many of these problems should be easy to translate into the conformal setting and perhaps something intelligent can be said regarding the Plücker images of such congruences. As has already been said, much of the analysis in this chapter can be repeated in the context of Lie sphere geometry [22]. It would be interesting to do this analysis rather than deducing results by hand-waving or an appeal to Lie's line-sphere correspondence: indeed since real spheres are mapped to complex lines under Lie's transformation we cannot prove theorems about real sphere congruences this way, the correspondence merely helps us decide what we would like to prove. In particular, we would like a concrete geometric idea of how the Laplace transforms of a sphere congruence relate—is there a relation between the curvatures of the corresponding spheres for instance? Can we view Laplace transforms of sphere congruences entirely in terms of their enveloping surfaces? Do the conformal coordinates x, y arising from a sphere congruence have any geometric meaning on the enveloping surfaces? We know that the curvature of the normal bundle vanishes iff we have a Ribaucour congruence, so does the curvature of the normal bundle measure how close the curvature lines are to coinciding in the same way that the Weingarten invariant measures the difference between asymptotic lines on focal surfaces? Can we see the 'R-invariant' and Laplace invariants of a sphere congruence—defined in terms of the curvature of the normal bundle and the Willmore density—directly in terms of its enveloping surfaces. One possible approach to this could be via a second real form of the complex Klein correspondence $\mathfrak{su}(2,2) \cong \mathfrak{so}(4,2)$: this should yield a 4-dimensional description of sphere congruences. Ferapontov [30] makes use of this approach. Finally, is there a sensible definition of the dual to a sphere congruence in S^3?

Chapter 3

Loop Groups and Simple Factors

3.1 Introduction

In this chapter we discuss various subgroups of the loop group $\mathcal{G} := \{g : \mathrm{dom}(g) \to G^{\mathbb{C}} \text{ holomorphic}\}$ where $\mathrm{dom}(g)$ is $\mathbb{P}^1 = \mathbb{P}(\mathbb{C})$ minus a finite set of points and $G^{\mathbb{C}}$ is the complexification of a compact Lie group: in particular we prove a version of the Birkhoff decomposition which allows us to build a local dressing action of the subgroup of negative loops $\mathcal{G}^-$ on the subgroup of positive loops $\mathcal{G}^+$. The theory is stated in such a manner as to apply in many situations, namely when we restrict to loops that are real ($\overline{g(z)} = g(\bar{z})$) with respect to some real form of $G^{\mathbb{C}}$ and/or twisted ($\tau g(z) = g(\omega z)$)[1] by some finite order automorphism τ of $G^{\mathbb{C}}$. Following Uhlenbeck [60] we look for the *simple factors* in $\mathcal{G}^-$ for which the dressing action is most easily calculated: calculation of the action depends only on the values of the inputs and not on their derivatives. For loops which are either untwisted, or twisted by an involution, we observe that the simple factors are classified by at most two complex scalars and a complementary pair of parabolic subalgebras of $\mathfrak{g}^{\mathbb{C}}$, the Lie algebra of $G^{\mathbb{C}}$. Moreover we calculate the dressing action of all simple factors in these cases and obtain a theorem of the Bianchi permutability of dressing transforms by simple factors. As a coda we discuss the existence of simple factors with respect to a higher order twisting τ. In particular a simple Lie algebra admits a *Coxeter automorphism* (of order the height of the maximal root plus 1). Terng–Uhlenbeck [56] provide an example of dressing by simple elements in $\mathrm{SL}(n)$, twisted with by the Coxeter automorphism (of order n). The construction of their simple elements may be described by a general theory applicable to any simple Lie algebra. We apply this construction to the orthogonal groups and find the simplest possible elements of $\mathcal{G}^-$ with which one can dress. It is seen however that the simplest elements for the orthogonal groups have a different flavour to those for the special linear group: in $\mathrm{SL}(n)$ we have fractional linear simple

[1] ω is a root of unity of the same order as τ.

elements, while in $SO(n)$ the factors are at simplest quadratic fractional. The lack of linearity is enough to cause the proof of the dressing action in $SL(n)$ to fail when one attempts to translate the argument to general groups.

3.2 The Birkhoff Factorisation Theorem

The theory of dressing actions on loop groups hinges on the Birkhoff decomposition. His theorem was discovered during his investigations of non-linear ODE's of the form $\frac{df}{dz} = G(z) \cdot f(z)$ where $G : \mathbb{C} \to GL_n(\mathbb{C})$ is holomorphic except for a simple pole at the origin. Birkhoff [4] proved that the solutions f to *most* equations of the above form can be multiplied by a holomorphic map into $GL_n(\mathbb{C})$ such that the transform $\tilde{f}$ satisfies a new equation of the form $\frac{d\tilde{f}}{dz} = z^{-1}\tilde{G} \cdot \tilde{f}$, where $\tilde{G}$ is a constant matrix. We will use the more modern version of Birkhoff's theorem to do much the same thing, where $GL_n(\mathbb{C})$ is replaced by a complex Lie group $G^{\mathbb{C}}$. Before considering the theorem itself, we must properly define its ingredients.

Definition 3.1
A loop *is a holomorphic map* $g : \mathrm{dom}(g) \subset \mathbb{P}^1 \to G^{\mathbb{C}}$ *where* $G^{\mathbb{C}}$ *is a complex Lie group and* $\mathrm{dom}(g)$ *is the complement of a finite subset of* $\mathbb{P}^1$. *The set of loops* $\mathcal{G}$ *forms a group under pointwise multiplication and is referred to as a* loop group.

In most texts a loop is a map $g : S^1 \to G^{\mathbb{C}}$, the set of such maps being usually denoted ΛG. Since $g \in \mathcal{G}$ is holomorphic and therefore uniformly convergent on any compact subset of $\mathrm{dom}(g)$ it follows that the compact open topology is a topology on $\mathcal{G}$: this is good enough to allow us to talk about open subsets of $\mathcal{G}$, but not enough to show that $\mathcal{G}$ is a manifold and therefore an infinite-dimensional Lie group. For this one must introduce a Frechet topology, the construction of which would take us too far afield. Two subgroups of $\mathcal{G}$ feature in the factorisation theorem; the positive and negative loops respectively.

Definition 3.2
The group of positive loops in $G^{\mathbb{C}}$ *is denoted*

$$\mathcal{G}^+ := \{g \in \mathcal{G} : \mathbb{C} \subset \mathrm{dom}(g)\},$$

whilst the negative loops similarly form a group:

$$\mathcal{G}^- := \{g \in \mathcal{G} : \infty \in \mathrm{dom}(g)\}.$$

The crucial point to take from what follows is that the intersection of these two groups is nothing but a copy of $G^{\mathbb{C}}$ itself, for $\mathcal{G}^+ \cap \mathcal{G}^-$ is the set of maps which are holomorphic on all of $\mathbb{P}^1$. Liouville's theorem assures us that all such maps are constant. To make full use of this observation it is customary to normalise one or both loop groups at a point. Consequently we write $\mathcal{G}^+_*$ for the positive loops satisfying $g(0) = \mathrm{Id}$. The intersection $\mathcal{G}^+_* \cap \mathcal{G}^-$ consists of just the identity map.

Theorem 3.3 (Birkhoff factorisation)
Let $G^{\mathbb{C}}$ be the complexification of a compact Lie group. The product set $\mathcal{G}^+_*\mathcal{G}^-$ is a dense open subset of the identity component of $\mathcal{G}$ while the multiplication map $\mathcal{G}^+_* \times \mathcal{G}^- \to \mathcal{G}$ is a diffeomorphism onto this set.

Since we have not shown that $\mathcal{G}$ is a manifold, we will only prove the first claim. Note however that the Birkhoff theorem of Pressley–Segal [49] applied to the restriction of $\mathcal{G}$ to all loops defined on a circle S^1 says that the multiplication map $\mathcal{G}^+_*|_{S^1} \times \mathcal{G}^-|_{S^1} \to \mathcal{G}|_{S^1}$ is a diffeomorphism onto the dense open subset $\mathcal{G}^+_*|_{S^1}\,\mathcal{G}^-|_{S^1}$ of the identity component of $\mathcal{G}|_{S^1}$. This is not however quite enough for the theorem. That multiplication is smooth is required in Chapter 4 where we consider smooth maps of some manifold $\Sigma \to \mathcal{G}$ and require that the products and inverses of such maps are smooth.

Proof Note firstly that the decomposition $g = h^+h^-$ ($h^\pm \in \mathcal{G}^\pm_{(*)}$), if it exists, is necessarily unique by Liouville:

$$h_1^+h_1^- = h_2^+h_2^- \Longrightarrow (h_2^+)^{-1}h_1^+ = h_2^-(h_1^-)^{-1} \in \mathcal{G}^+_* \cap \mathcal{G}^- = \{\mathrm{Id}\}.$$

Suppose $g \in \mathcal{G}$ is a loop. Then there exists an $r > 0$ such that g is holomorphic on $\{z \in \mathbb{C} : |z| > r\}$. Apply the usual statement of Birkhoff's theorem (e.g. in Pressley–Segal [49]) to g restricted to the circle S^1_r: there exists a dense open set of the identity component of the loop group $\{g : S^1_r \to G^{\mathbb{C}}\}$ such that for $g|_{S^1_r}$ in this set there exist unique maps h_r^+, h_r^- such that $g|_{S^1_r} = h_r^+h_r^-$ where h_r^+, h_r^- are holomorphic on $|z| < r$ and $r < |z| \leq \infty$ respectively, continuous up to $|z| = r$ and $h_r^+(0) = 1$. Extend h_r^+, h_r^- to continuous functions on $\mathbb{C}$, $\mathrm{dom}(g) \cup \infty$ respectively, holomorphic away from S^1_r by

$$h^+ := \begin{cases} h_r^+ & |z| \leq r, \\ g(h_r^-)^{-1} & |z| \in [r, \infty), \end{cases} \qquad h^- := \begin{cases} (h^+)^{-1}g & z \in \mathrm{dom}(g) \cap \mathbb{C}, \\ h_r^- & |z| \in [r, \infty], \end{cases}$$

with exceptionally $h^+(\infty) := g(\infty)h^-(\infty)^{-1}$ if $g(\infty)$ is defined. Using continuity across S^1_r to approximate contour integrals on curves across S^1_r by contours on either side of S^1_r we see that the integrals of $h^\pm$ over any closed loop near S^1_r is zero.[2] Morera's theorem (e.g. [50]) implies that $h^\pm$ are holomorphic across S^1_r and so we have built $h^\pm \in \mathcal{G}^\pm_{(*)}$ such that $g = h^+h^-$. By the uniqueness comment above it is clear that $h^\pm$ so defined are independent of whichever r we start with. This proves the first claim.

■

That $G^{\mathbb{C}}$ is the complexification of a compact G is required only to put ourselves in the setting of [49]. Complexifications of arbitrary G need not exist (the simply-connected covering group of $SL_2(\mathbb{R})$ being the standard example). If G is a compact Lie group we can invoke the Peter-Weyl theorem [40, 61] which says that any such G can be embedded in a unitary group (i.e. there exists a faithful linear representation). The complexification of G is then realised via $U_n^{\mathbb{C}} = GL_n(\mathbb{C})$ (just drop the $gg^* = \mathrm{Id}$ condition). For non-compact G one must work a little harder. We are saved since our later work is more concerned with Lie algebras than Lie groups. Indeed we will tend to work with real semisimple Lie algebras $\mathfrak{g}$. Any real semisimple $\mathfrak{g}$ possesses a Cartan decomposition (cf. Section 1.3) $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: the real Lie algebra $\mathfrak{g}_T := \mathfrak{k} \oplus i\mathfrak{p}$ is therefore compact and semisimple; a *compact dual* of $\mathfrak{g}$. It is clear that $\mathfrak{g}, \mathfrak{g}_T$ have the same complexification $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C} = \mathfrak{g}_T \otimes \mathbb{C}$. It is now straightforward to construct Lie groups satisfying the hypotheses of Birkhoff's theorem: since $\mathfrak{g}^{\mathbb{C}}$ is semisimple, $\mathrm{ad} : \mathfrak{g}^{\mathbb{C}} \to \mathrm{ad}(\mathfrak{g}^{\mathbb{C}})$ is an isomorphism and so $\mathfrak{g}^{\mathbb{C}}$ is the Lie algebra of $G^{\mathbb{C}} := \mathrm{Int}(\mathfrak{g}^{\mathbb{C}})$, the semisimple, centre-free Lie group of inner automorphisms of $\mathfrak{g}^{\mathbb{C}}$. It is clear that $G := \mathrm{Int}(\mathfrak{g})$, $G_T := \mathrm{Int}(\mathfrak{g}_T)$ are real forms of $G^{\mathbb{C}}$ and that G_T is compact. In later discussions where a semisimple Lie algebra, but not a Lie group, is given it will be assumed that $G, G^{\mathbb{C}}$ are diffeomorphs of the above groups of automorphisms.

The factorisation theorem still holds if we restrict to negative loops normalised at zero: $\mathcal{G}^-_* := \{g \in \mathcal{G}^- : 0 \in \mathrm{dom}(g), g(0) = \mathrm{Id}\}$. Some authors (e.g. [9, 56]) prefer to work with zero and ∞ swapped: i.e. no restriction is made to positive loops, while negative loops are normalised at ∞. When it is not important to stress where the normalisation occurs we will tend to drop asterisks and assume that at least one normalisation has been performed so that $\mathcal{G}^\pm$ satisfy a factorisation theorem.

The purpose of Theorem 3.3 is to build a local dressing action of negative loops on positive, or vice versa. Suppose for a moment that $\mathcal{G}^\pm$ are subgroups of some $\mathcal{G}$ such that $\mathcal{G} = \mathcal{G}^+\mathcal{G}^-$ and $\mathcal{G}^+ \cap \mathcal{G}^- = \{\mathrm{Id}\}$: it is clear that given any $g_\pm \in \mathcal{G}^\pm$ there exist unique $h_\pm \in \mathcal{G}^\pm$ such that $g_-g_+ = h_+h_-$ and so we may define an action # of $\mathcal{G}^-$ on

[2] The existence of a faithful linear representation of $G^{\mathbb{C}}$ allows us make sense of contour integrals of the group-valued functions $h^\pm$.

$\mathcal{G}^+$ by

$$g_- \# g_+ := h_+ = g_- g_+ h_-^{-1}.$$

Lemma 3.4
 # really is an action.

Proof Let $g_1, g_2 \in \mathcal{G}^-$, $g_+ \in \mathcal{G}^+$ and suppose that $g_1 \#(g_2 \# g_+)$ and $(g_1 g_2) \# g_+$ are defined. There exist unique $h_1, h_2, h_{12} \in \mathcal{G}^-$ such that,

$$\left.\begin{array}{l} g_1 \#(g_2 \# g_+) = g_1 g_2 g_+ h_2^{-1} h_1^{-1} \\ (g_1 g_2) \# g_+ = g_1 g_2 g_+ h_{12}^{-1} \end{array}\right\} \Longrightarrow [g_1 \#(g_2 \# g_+)] h_2 h_1 = [(g_1 g_2) \# g_+] h_{12},$$

which, by $\mathcal{G}^+ \cap \mathcal{G}^- = \{\mathrm{Id}\}$, implies

$$g_1 \#(g_2 \# g_+) = (g_1 g_2) \# g_+, \qquad h_2 h_1 = h_{12}.$$

■

In our case $\mathcal{G}^+\mathcal{G}^-$ is open in $\mathcal{G}$ and so we only have an action when $g_- g_+ \in \mathcal{G}^+\mathcal{G}^-$. However, by two applications of the Birkhoff theorem, both $\mathcal{G}^+\mathcal{G}^-$ and $\mathcal{G}^-\mathcal{G}^+$ are open and dense in the identity component of $\mathcal{G}$ and so a generic map $g = g_- g_+$ has a unique factorisation $g = \hat{g}_+ \hat{g}_-$. Moreover by the openness we have that if $g_- \# g_+$ is defined, then for minor perturbations of either ingredient the action is still defined. The action is therefore only *locally* defined.

Definition 3.5
The dressing action *of $\mathcal{G}^-$ on $\mathcal{G}^+$ is the local action $g_- \# : g_+ \mapsto \hat{g}_+$.*

3.3 Reality and Twisting Conditions

It is often desirable to impose extra restrictions on the loop groups $\mathcal{G}^\pm$: we will consider loops satisfying a reality condition with respect to some real form of $G^{\mathbb{C}}$ and loops that are twisted with respect to a finite order automorphism of $G^{\mathbb{C}}$. The question is then whether these conditions are preserved by the above dressing action.

Reality: Suppose G is a (not necessarily compact) real form of $G^{\mathbb{C}}$. G is synonymous with a *conjugation*: an anti-holomorphic involution $\bar{\ }$ of $G^{\mathbb{C}}$ with fixed set G.[3]

Definition 3.6
Fix a real form G of $G^{\mathbb{C}}$. A loop $g \in \mathcal{G}$ is real *if $g(\bar{z}) = \overline{g(z)}$, $\forall z \in \mathrm{dom}(g)$.*

[3]Not to be confused with conjugation in $\mathbb{C}$: the context will make the meaning clear.

It is clear that real loops form a group $\mathcal{G}_r$ since conjugation is an automorphism of $G^{\mathbb{C}}$. We similarly have subgroups of real positive and negative loops $\mathcal{G}_r^{\pm}$.

Proposition 3.7
The Birkhoff decomposition restricts to real loops.

Proof Suppose $g = g_+g_-$ is the Birkhoff decomposition of a real loop g. Then

$$g_+(\bar{z})g_-(\bar{z}) = \overline{g_+(z)g_-(z)} \implies g_+(z)^{-1}\overline{g_+(\bar{z})} = \overline{g_-(\bar{z})}g_-(z)^{-1}.$$

$\overline{g_+(\bar{z})}$ is holomorphic since any conjugation must send holomorphic maps to anti-holomorphic maps. Both sides of the above are therefore holomorphic loops and so live in the groups $\mathcal{G}^{\pm}$ respectively. The usual Liouville argument forces both $g_{\pm} \in \mathcal{G}_r^{\pm}$. ∎

It follows that the dressing action respects reality conditions: if g_-, g_+ are real then so is $g_- \# g_+$.

Twisting: Let τ be an order n holomorphic[4] automorphism of $G^{\mathbb{C}}$ and let ω be a primitive n^{th} root of unity.

Definition 3.8
A loop $g \in \mathcal{G}$ is twisted with respect to τ if

$$\tau g(z) = g(\omega z), \ \forall z \in \operatorname{dom}(g).$$

It is clear that twisted loops form subgroups $\mathcal{G}_\tau, \mathcal{G}_\tau^{\pm}$. By a similar argument to Proposition 3.7 it can also be seen that the dressing action respects twisting.

We will also work with loops that are both real and twisted, i.e. $\mathcal{G}_{r,\tau}$. In such cases we will additionally assume that τ commutes with conjugation across the real form, so that τ restricts to an automorphism of G. Twisting and reality conditions play little more than a motivational role in this chapter, their introduction is mainly in preparation for dressing $\mathfrak{p}$-flat maps in Chapter 4. In the following discussion one can liberally replace the groups $\mathcal{G}^{\pm}$ with any combination of based $(*)$, twisted (τ), or real (r) loops. When considering twisted loops we will be predominantly concerned with involutive τ rather than higher order—if not mentioned then the reader should assume that τ has order 2. The exception to this is Section 3.8 where we consider certain τ of higher order.

[4] I.e. $\tau\mathcal{G} \subset \mathcal{G}$.

3.4 Simple Factors

Partly in preparation for Chapter 4 when we will pointwise dress maps $\Phi : \Sigma \to \mathcal{G}^+$, and partly so as to distinguish more clearly the 'dresser' $g_- \in \mathcal{G}^-$ from the 'dressee' $g_+ \in \mathcal{G}^+$ we will denote positive loops by Φ from now on.

As yet, no indication has been given as to how to perform the factorisation $g_-\Phi = \hat{\Phi}\hat{g}_-$. In general this is a Riemann-Hilbert problem and therefore extremely difficult to solve. Our goal is to find special g_- for which the factorisation is not only calculable, but calculable by algebra alone. Such g_- for which the dependence on Φ is algebraic are loosely referred to in the literature as *simple factors* [9, 11, 56], although a concrete definition more appropriate to our uses will be given below.

Let us be more explicit: if $g_- \not\equiv \mathrm{Id}$ then there must be some points where g_- has a pole, or fails to be invertible. If there are no extra conditions on the loops then the simplest choice is to let g_- be rational with a pole at some $\alpha \in \mathbb{C}$. Suppose however that we have a reality condition and an order 2 twisting condition: if α is a pole, then so are $-\alpha, \overline{\alpha}$. Since $\alpha \neq 0, \infty$, the simplest choice is to take $\alpha \in \mathbb{R}^\times \cup i\mathbb{R}^\times$ and let g_- be rational on $\mathbb{P}^1$ with poles at $\pm\alpha$. The following is a theorem of Burstall [11] adapted from loop group factoring results of Pressley–Segal [49].

Theorem 3.9
If G is a compact Lie group and $g_- \in \mathcal{G}^-_{*,r,\tau}$ rational on $\mathbb{P}^1$ with poles at $\pm\alpha$ ($\alpha \in i\mathbb{R}^\times$), then the action of g_- on $\Phi \in \mathcal{G}^+$ can be computed in terms of g_- and a finite jet[5] of Φ at α. Moreover the dependence of $g_- \# \Phi$ on Φ is algebraic (depends only on the 0-jet $\Phi(\alpha)$) iff $g_-(z) = \gamma(\frac{\alpha - z}{\alpha + z})$ where $\mathrm{Ad}\,\gamma : \mathbb{C}^\times \to GL(\mathfrak{g}^{\mathbb{C}})$ is a homomorphism with simple poles.

Since all our groups $G^{\mathbb{C}}$ are adjoint we say that $g_- \in \mathcal{G}^-_*$ is a *simple factor* if $g_-(z) = \gamma(t_{\alpha,\beta}(z))$ where $t_{\alpha,\beta} = \frac{1-\alpha^{-1}z}{1-\beta^{-1}z}$ for some $\alpha, \beta \in \mathbb{C}^\times$ and γ is a homomorphism with simple poles. As we shall show in Section 3.6, the action of such a g_- can be calculated explicitly, the result depending only on the values of Φ at α, β. It will furthermore be shown that if g_- is a simple factor then so is the dressed negative loop $\hat{g}_-$, the only thing that changes being the homomorphism γ. The above property is very special, for if we complicate matters only slightly and let γ have double poles, then not only does $\hat{g}_-$ fail to depend algebraically on Φ, but $\hat{\gamma} := \hat{g}_- \circ t^{-1}_{\alpha,\beta}$ is not even a homomorphism.

The choice of fractional linear $t_{\alpha,\beta}$ depends on context. If we assume that $g_- \in \mathcal{G}^-_{*,\tau}$, then we quickly see that $t(z) = \frac{\alpha - z}{\alpha + z}$ where $\alpha \in \mathbb{C}^\times$. Similarly $g_- \in \mathcal{G}^-_{*,r}$ forces $t(z) = \frac{1+\alpha z}{1+\beta z}$ where either α, β are both real and non-zero, or $\beta = \overline{\alpha}$. Applying both twisting

[5] $\Phi(\alpha)$ and a finite number of its derivatives $\left.\frac{\mathrm{d}^n\Phi}{\mathrm{d}z^n}\right|_{z=\alpha}$.

and reality imposes the conditions of the theorem except that now $\alpha \in \mathbb{R}^\times \cup i\mathbb{R}^\times$. The restriction to imaginary α in the theorem is because $g_-|_\mathbb{R}$ is a smooth map into a real compact group and so cannot have any poles. For non-compact G there is no such restriction.

If instead we normalise g_- at ∞, it turns out that the Bäcklund transforms of Terng–Uhlenbeck [56] fit into this framework: suppose $G^\mathbb{C} = \mathrm{GL}_n(\mathbb{C})$ and let $g_- \in \mathcal{G}^-$ be rational with a pole at some $\alpha \in \mathbb{C}$ and $g_-(\infty) = \mathrm{Id}$. Any such g_- is the product of maps of 'simplest type' ([60] theorem 5.4)[6]

$$h_{\alpha,\beta,\pi}(z) = \mathrm{Id} + \frac{\alpha - \beta}{z - \alpha}\pi'$$

where $\pi = \mathrm{Proj}_V$ and $\pi' = \mathrm{Proj}_W$ such that $\mathbb{C}^n = V \oplus W$. It is easy to see that $h^{-1}_{\alpha,\beta,\pi} = h_{\beta,\alpha,\pi}$ and so $\mathrm{Ad}\, h$ has eigenspaces $W^* \otimes V$, $(V^* \otimes V) \oplus (W^* \otimes W)$, $V^* \otimes W$ with $t_{\alpha,\beta}(z) = \frac{z-\alpha}{z-\beta}$.

The geometry of simple factors is that of the eigenspaces of γ: the classification of simple factors and the eventual calculation of their dressing action depends on the observation that that $\gamma(t)$ decomposes $\mathfrak{g}^\mathbb{C}$ into three eigenspaces $\mathfrak{g}_+, \mathfrak{g}_0, \mathfrak{g}_-$ with eigenvalues $t, 1, t^{-1}$ respectively, and that the two subalgebras

$$\mathfrak{q} := \mathfrak{g}_+ \oplus \mathfrak{g}_0, \qquad \mathfrak{r} := \mathfrak{g}_0 \oplus \mathfrak{g}_-$$

are complementary parabolic subalgebras (Section 1.3) with Abelian nilradicals.

3.5 Classification of Simple Factors

Recall the definition of simple factor inspired by Theorem 3.9. We show that every simple factor determines and is determined by a complementary pair of parabolic subalgebras with Abelian nilradicals and in so doing classify the simple factors. Given the existence of simple factors, the discussion of Section 1.3 shows that the choice of Lie algebra $\mathfrak{g}^\mathbb{C}$ is severely restricted: the decomposition of $\mathfrak{g}^\mathbb{C}$ into simple ideals must contain one of the algebras $\mathfrak{sl}, \mathfrak{so}, \mathfrak{sp}, \mathfrak{e}_6, \mathfrak{e}_7$, as these are the only simple Lie algebras possessing simple roots of weight 1.

Given a simple factor $g_- = \gamma \circ t_{\alpha,\beta}$, define

$$\xi = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=1} \gamma(t) \in \mathfrak{g}^\mathbb{C}.$$

[6] One may make the entire discussion semisimple by dividing through by $\det g_-$.

$\operatorname{Ad}\gamma$ is a homomorphism, so the following diagram commutes:

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\operatorname{ad}\xi"] \arrow[d, "\exp"'] & \mathfrak{gl}(\mathfrak{g}^{\mathbb{C}}) \arrow[d, "\exp"'] \\ \mathbb{C}^{\times} \arrow[r, "\operatorname{Ad}\gamma"] & GL(\mathfrak{g}^{\mathbb{C}}) \end{tikzcd}$$

I.e. $\operatorname{Ad}\gamma(e^z) = \exp(z \operatorname{ad}\xi)$. Since e^z is periodic we have

$$\operatorname{Ad}\gamma(e^{2\pi i}) = \mathrm{Id} = \exp(2\pi i \operatorname{ad}\xi). \tag{3.1}$$

Let the Jordan decomposition[7] of $2\pi i \operatorname{ad}\xi$ be $S + N$. It follows that $\exp(N) = \exp(-S)$ and so N is diagonalisable, forcing $N = 0$. $\operatorname{ad}\xi$ is therefore diagonalisable, while (3.1) insists on integer eigenvalues. The simple poles condition makes these eigenvalues precisely $\pm 1, 0$. Denoting the eigenspaces of $\operatorname{ad}\xi$ by $\mathfrak{g}_{\pm}, \mathfrak{g}_0$ and projection maps by

$$\pi_j : \mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_- \to \mathfrak{g}_j,$$

allows us to write

$$\operatorname{Ad}\gamma(t) = t\pi_+ + \pi_0 + t^{-1}\pi_-, \qquad \operatorname{ad}\xi = \pi_+ - \pi_-. \tag{3.2}$$

Proposition 3.10
$\mathfrak{q} := \mathfrak{g}_+ \oplus \mathfrak{g}_0$, $\mathfrak{r} := \mathfrak{g}_0 \oplus \mathfrak{g}_-$ are complementary parabolic subalgebras of $\mathfrak{g}^{\mathbb{C}}$ with nilradicals $\mathfrak{g}_{\pm}$ respectively.

Proof That $\mathfrak{g}^{\mathbb{C}}$ is a graded algebra with grading element ξ is easily seen by applying the Jacobi identity to $\operatorname{ad}\xi[\,\cdot\,,\,\cdot\,]$. In particular $\operatorname{ad}\xi[\mathfrak{g}_+, \mathfrak{g}_+] = 2[\mathfrak{g}_+, \mathfrak{g}_+]$ and so $\mathfrak{g}_+$ (resp. $\mathfrak{g}_-$) is Abelian. Consider the Killing form of $\mathfrak{g}^{\mathbb{C}}$:

$$B(p, q) = \operatorname{tr}(\operatorname{ad}(p)\operatorname{ad}(q)) = \sum_{\text{any basis } \{g_i\}} \operatorname{coeff}([p, [q, g_i]], g_i).$$

Evaluating on basis elements of $\mathfrak{g}_0, \mathfrak{g}_{\pm}$ makes it easy to see that the nilradicals are $\mathfrak{q}^{\perp} = \mathfrak{g}_+$, $\mathfrak{r}^{\perp} = \mathfrak{g}_-$ as required.

■

A simple factor $g_- = \operatorname{Ad}\gamma$ gives rise to a pair of complementary parabolic subalgebras simply by differentiating γ to find the canonical element. The converse is also true: given a pair of complementary parabolic subalgebras with canonical element ξ,

[7] $S + N$ is the unique semisimple (diagonalisable) + nilpotent decomposition such that $[S, N] = 0$, see e.g. [38].

define $\gamma(t) = \exp((\ln t)\xi)$. $\mathrm{Ad}\,\gamma$ is then a homomorphism $\mathbb{C}^\times \to \mathrm{Ad}(\mathfrak{g}^{\mathbb{C}})$ with simple poles.

In the absence of twisting or reality conditions the above is the whole story: a simple factor is exactly a choice of a pair of complementary parabolic subalgebras and a fractional linear t. In the presence of conditions however, the choice of parabolic subalgebras is restricted.

- The reality condition depends on the choice of $t_{\alpha,\beta}$. In terms of the decomposition (3.2) and supposing that the centre of G is trivial, the reality condition reads

$$t(\bar{z})\pi_+ + \pi_0 + t(\bar{z})^{-1}\pi_- = \overline{t(z)}\overline{\pi}_+ + \overline{\pi}_0 + \overline{t(z)}^{-1}\overline{\pi}_-.$$

Given our two possibilities for $t_{\alpha,\beta}$ we see that

$$t(z) = \frac{1+\alpha z}{1+\beta z}, \ \alpha,\beta \in \mathbb{R}^\times \Longrightarrow \overline{\pi}_+ = \pi_+ \Longrightarrow \bar{\xi} = \xi,$$
$$t(z) = \frac{1+\alpha z}{1+\bar{\alpha} z}, \ \alpha \in \mathbb{C}^\times \Longrightarrow \overline{\pi}_+ = \pi_- \Longrightarrow \bar{\xi} = -\xi.$$

- For the twisting condition there are no choices:

$$\mathrm{Ad}\,\tau(\gamma) = (\mathrm{Ad}\,\gamma)^{-1} \Longrightarrow \mathrm{ad}\,\tau(\xi) = -\,\mathrm{ad}\,\xi \Longrightarrow \tau(\xi) = -\xi.$$

More is true, for $\mathrm{ad}\,\tau(\xi) = \tau \circ \mathrm{ad}\,\xi \circ \tau^{-1} \Rightarrow \tau\mathfrak{g}_+ = \mathfrak{g}_-$, $\tau\mathfrak{g}_0 = \mathfrak{g}_0$, implying that the complementary parabolic subalgebra $\mathfrak{r}$ is in fact $\tau\mathfrak{q}$.[8]

We have established the following classification theorem:

Theorem 3.11
The simple factors in the loop group $\mathcal{G}^-_{*,(r,\tau)}$ are classified by a choice of constants α,β and a pair of complementary parabolic subalgebras $\mathfrak{q},\mathfrak{r}$ with canonical element ξ, satisfying the following conditions, dependent on whether we have a reality condition, twisting or both.

N/C $\quad p_{\alpha,\beta,\mathfrak{q},\mathfrak{r}}(z) = \exp\left(\ln\left(\frac{1-\alpha^{-1}z}{1-\beta^{-1}z}\right)\xi\right), \quad \alpha,\beta \in \mathbb{C}^\times,$

Reality $\quad p_{\alpha,\beta,\mathfrak{q},\mathfrak{r}}(z) = \exp\left(\ln\left(\frac{1-\alpha^{-1}z}{1-\beta^{-1}z}\right)\xi\right), \quad \alpha,\beta \in \mathbb{R}^\times,\ \bar{\xi} = \xi,$

[8]The observation that τ preserves conjugacy classes of parabolic subalgebras (Lemma 5.5) says that there are *no* real & twisted simple factors when the real conjugacy class of $\mathfrak{q}$ is a non-self-dual symmetric R-space (Section 5.3).

$$p_{\alpha,\mathfrak{q}}(z) = \exp\left(\ln\left(\frac{1-\alpha^{-1}z}{1-\overline{\alpha}^{-1}z}\right)\xi\right), \quad \alpha \in \mathbb{C}^\times, \overline{\xi} = -\xi,$$

Twisting $$p_{\alpha,\mathfrak{q}}(z) = \exp\left(\ln\left(\frac{\alpha - z}{\alpha + z}\right)\xi\right), \quad \alpha \in \mathbb{C}^\times, \tau\xi = -\xi,$$

Reality & twisting $$p_{\alpha,\mathfrak{q}}(z) = \exp\left(\ln\left(\frac{\alpha - z}{\alpha + z}\right)\xi\right), \quad \tau\xi = -\xi, \begin{cases} \overline{\xi} = \xi, & \alpha \in \mathbb{R}^\times, \\ \overline{\xi} = -\xi, & \alpha \in i\mathbb{R}^\times. \end{cases}$$

The simple factors are well-defined: ambiguities in the choice of logarithm are irrelevant since $\exp(2\pi i\xi)$ *is in the centre of G by (3.1), which is trivial by the the assumptions of page 58.*

The labels on the simple factors are chosen for a reason. Depending on the existence of twisting and reality conditions, the α, β reveal the poles of p. We stress only $\mathfrak{q}$ for twisted maps since $\mathfrak{r} = \tau\mathfrak{q}$ is implied, while $\mathfrak{r} = \overline{\mathfrak{q}}$ for the second real example. The choice of labelling simple factors with parabolic subalgebras instead of canonical elements is because the dressing action of simple factors will be seen to be easy to write down in terms of $\mathfrak{q}, \mathfrak{r}$, whereas the resulting action on canonical elements is highly non-trivial.

3.6 Dressing by Simple Factors

Having classified the simple factors in the previous section, it remains to calculate their dressing action on maps $\Phi \in \mathcal{G}^+$ and to show that it indeed depends algebraically on Φ, in the sense of Theorem 3.9. To facilitate easier calculations we move the poles of our simple factor to $0, \infty$ via the fractional linear transformation $t_{\alpha,\beta}(z) = \frac{1-\alpha^{-1}z}{1-\beta^{-1}z}$ and work with the maps $\gamma = g_- \circ t_{\alpha,\beta}^{-1}$. The map $\Phi : \Sigma \to \mathcal{G}^+$ that we wish to dress now becomes $E := \Phi \circ t_{\alpha,\beta}^{-1}$, which is holomorphic on $\mathbb{P}^1 \setminus \{\alpha, \beta\}$.

Theorem 3.12
Let $\mathfrak{q}, \mathfrak{r}$ *be complementary parabolic subalgebras of* $\mathfrak{g}^\mathbb{C}$ *with Abelian nilradicals and canonical element* ξ *and let* $G^\mathbb{C} := \mathrm{Int}(\mathfrak{g}^\mathbb{C})$. *Define*

$$\gamma_{\mathfrak{q},\mathfrak{r}}(z) = \exp((\ln z)\xi),$$

so that

$$\gamma(z) = z\pi_+ + \pi_0 + z^{-1}\pi_-, \qquad \pi_j : \mathfrak{g}^\mathbb{C} \to \mathfrak{g}_j.$$

Let E be a map into $G^\mathbb{C}$ *holomorphic near* $0, \infty$. *Suppose p is a map into* $G^\mathbb{C}$, *holo-*

morphic on $\mathbb{C}^\times$ *with* $p(1) = \mathrm{Id}$. *Then* $\hat{E} := \gamma E p^{-1}$ *is holomorphic and invertible at* $0, \infty \iff p = \gamma_{\mathfrak{q}', \mathfrak{r}'}$ *where*

$$\hat{\mathfrak{q}} := E^{-1}(0)\mathfrak{q}, \qquad \hat{\mathfrak{r}} := E^{-1}(\infty)\mathfrak{r}$$

are complementary.

The restriction to the adjoint group $\mathrm{Int}\,\mathfrak{g}^{\mathbb{C}}$ is not strictly necessary but does make the notation in the proof a little easier. Moreover the main application we will make of the theorem (in Chapter 4) does not require explicit knowledge of the group, only of the algebra, so that in the abstract we lose nothing. Even when it comes to examples where G is easier to work with than $\mathrm{Ad}(\mathfrak{g})$, it tends to be easier to invoke the theorem before worrying about how the isomorphism $G/Z(G) \cong \mathrm{Ad}(\mathfrak{g})$ affects solutions.

Proof 1. Expand E in formal power series about zero and ∞:

$$E(\lambda) = \sum_{k \geq 0} \lambda^k E_k, \qquad E(\mu^{-1}) = \sum_{k \geq 0} \mu^k F_k,$$

and observe that

$$E_1 E_0^{-1} = \left.\frac{\partial}{\partial \lambda}\right|_{\lambda=0} E(\lambda) E_0^{-1} \in \mathrm{ad}\,\mathfrak{g}^{\mathbb{C}},$$

and similarly for $F_1 F_0^{-1}$.

2. By a familiar Liouville argument we see that if $\hat{E}, p$ exist with the properties claimed, they are unique: if not

$$\hat{E}_1 p_1 = \hat{E}_2 p_2 \Longrightarrow \hat{E}_1 \hat{E}_2^{-1} = p_1^{-1} p_2,$$

where the left side is holomorphic near $0, \infty$, and the right side is holomorphic everywhere else; since $p(1) = \mathrm{Id}$, Liouville's theorem says that both sides are the identity.

3. Define $\hat{\mathfrak{q}}, \hat{\mathfrak{r}}$ as in the theorem and suppose that they are not complementary: i.e. $\exists$ non-zero $v = E_0^{-1}(g_0 + g_+) = F_0^{-1} g_-$ for some $g_j \in \mathfrak{g}_j$. Suppose further that $\exists \hat{E}, p$ such that $\gamma E = \hat{E} p$ with $\hat{E}$ holomorphic and invertible at zero and infinity. p has the same poles as γ at $0, \infty$ and $p(1) = \mathrm{Id}$, so $\exists a, b, c \in \mathrm{End}(\mathfrak{g}^{\mathbb{C}})$ s.t.

$$p(\lambda) = \lambda a + b + c\lambda^{-1}, \qquad a + b + c = \mathrm{Id}.$$

Expanding $\gamma E, \hat{E}p$ about $0, \infty$ and equating coefficients gives us

$$\begin{aligned}
\gamma E &= \lambda^{-1}\pi_- E_0 + \pi_0 E_0 + \pi_- E_1 + \lambda(\cdots) \\
&= \mu^{-1}\pi_+ F_0 + \pi_0 F_0 + \pi_+ F_1 + \mu(\cdots), \\
\hat{E}p &= \lambda^{-1}\hat{E}_0 c + \hat{E}_0 b + \hat{E}_1 c + \lambda(\cdots) \\
&= \mu^{-1}\hat{F}_0 a + \hat{F}_0 b + \hat{F}_1 a + \mu(\cdots), \\
\therefore\ a(v) &= \hat{F}_0^{-1}\pi_+ g_- = 0, \\
c(v) &= \hat{E}_0^{-1}\pi_-(g_0 + g_+) = 0, \\
v = b(v) &= \hat{F}_0^{-1}(\pi_0 g_- + \pi_+ F_1 F_0^{-1} g_-) = \hat{F}_0^{-1}\pi_+(F_1 F_0^{-1}) g_- = 0
\end{aligned}$$

by part 1, since $\mathrm{ad}(\mathfrak{g}^{\mathbb{C}})\mathfrak{g}_- \subset \mathfrak{g}_0 \oplus \mathfrak{g}_-$. This is a contradiction and so $\hat{\mathfrak{q}}, \hat{\mathfrak{r}}$ must be complementary for the dressing action to be defined.

4. Now suppose that $\hat{\mathfrak{q}}, \hat{\mathfrak{r}}$ are complementary. We can therefore define the homomorphism $\hat{\gamma} := \gamma_{\hat{\mathfrak{q}},\hat{\mathfrak{r}}}$. Since $\bar{\partial}(\gamma E \hat{\gamma}^{-1}) = 0$ we have $\hat{E}$ holomorphic at $0, \infty$ iff it is pole free at $0, \infty$. About zero we have

$$\begin{aligned}
\hat{E}(\lambda) &= (\lambda\pi_+ + \pi_0 + \lambda^{-1}\pi_-)\sum_{k\geq 0}\lambda^k E_k(\lambda^{-1}\hat{\pi}_+ + \hat{\pi}_0 + \lambda\hat{\pi}_-) \\
&= \lambda^{-2}\pi_- E_0 \hat{\pi}_+ + \lambda^{-1}(\pi_- E_0 \hat{\pi}_0 + \pi_- E_1 \hat{\pi}_+ + \pi_0 E_0 \hat{\pi}_+) \\
&\qquad + \text{holomorphic term.}
\end{aligned}$$

$\hat{E}$ is therefore holomorphic at zero iff the coefficients of $\lambda^{-1}, \lambda^{-2}$ in the above expansion are zero. However

$$\hat{\mathfrak{q}} = E_0^{-1}\mathfrak{q} \iff \hat{\mathfrak{g}}_+ = E_0^{-1}\mathfrak{g}_+ \iff \pi_- E_0 \hat{\pi}_+ = \pi_- E_0 \hat{\pi}_0 = \pi_0 E_0 \hat{\pi}_+ = 0.$$

For the final condition,

$$\begin{aligned}
\pi_- E_1 \hat{\pi}_+ = 0 &\iff E_1 \hat{\mathfrak{g}}_+ \subset \mathfrak{q} \iff E_1 E_0^{-1}\mathfrak{g}_+ \subset \mathfrak{q} \\
&\iff (E_1 E_0^{-1})\mathfrak{g}_+ \subset \mathfrak{q},
\end{aligned}$$

which is trivial since $\mathrm{ad}(\mathfrak{g}^{\mathbb{C}})\mathfrak{g}_+ \subset \mathfrak{q}$.

For invertibility at zero, apply the same argument to $\hat{E}^{-1} = \hat{\gamma}E^{-1}\gamma^{-1}$. To see holomorphicity and invertibility at ∞, repeat the above with the expansions about ∞: this time the required condition is $\hat{\mathfrak{r}} = F_0^{-1}\mathfrak{r} = E^{-1}(\infty)\mathfrak{r}$.

■

Corollary 3.13
Let $p_{\alpha,\beta,\mathfrak{q},\mathfrak{r}} \in \mathcal{G}_*^-$ be a simple factor and let Φ be a holomorphic map into $G^{\mathbb{C}}$ defined near α, β. Suppose further that $\Phi^{-1}(\alpha)\mathfrak{q}, \Phi^{-1}(\beta)\mathfrak{r}$ are a complementary pair of parabolic subalgebras. Then

- $\hat{p} := p_{\alpha,\beta,\Phi_\alpha^{-1}\mathfrak{q},\Phi_\beta^{-1}\mathfrak{r}} \in \mathcal{G}_*^-$,
- $\hat{\Phi} := p_{\alpha,\beta,\mathfrak{q},\mathfrak{r}}\Phi p^{-1}_{\alpha,\beta,\Phi_\alpha^{-1}\mathfrak{q},\Phi_\beta^{-1}\mathfrak{r}}$ *is holomorphic and invertible at α, β.*

Furthermore, if p, Φ satisfy reality/twisting conditions then so do $\hat{p}, \hat{\Phi}$.

This result is more general than we require in order to calculate the dressing action of simple factors on maps into $\mathcal{G}^+$: such generality is required for the permutability of such transforms in the next section.

Proof Composing part 3 of Theorem 3.12 with $t_{\alpha,\beta}$ shows that $\Phi_\alpha^{-1}\mathfrak{q}$, $\Phi_\beta^{-1}\mathfrak{r}$ are complementary iff $\hat{p}$ is well-defined and so the dressing action of $p_{\alpha,\beta,\mathfrak{q},\mathfrak{r}}$ is well-defined. Furthermore

$$\hat{p}(0) = \hat{\gamma}(1) = \mathrm{Id},$$

which gives the first claim. The second claim is just the final part of Theorem 3.12 after composition with $t_{\alpha,\beta}$.

There are two possible reality conditions. The first involves simple factors of the form $p_{\alpha,\beta,\mathfrak{q},\mathfrak{r}}$ where $\alpha, \beta \in \mathbb{R}^\times$ and $\mathfrak{q}, \mathfrak{r}$ are the complexifications of real parabolic subalgebras. Suppose that Φ is also real. Then

$$\overline{\Phi_\alpha^{-1}\mathfrak{q}} = \overline{\Phi_\alpha}^{-1}\overline{\mathfrak{q}} = \Phi_\alpha^{-1}\mathfrak{q}.$$

The $\mathfrak{r}$ condition is similar and so $\hat{p}$ is real. The second possibility has simple factors of the form $p_{\alpha,\mathfrak{q}}$ with poles at $\alpha, \overline{\alpha}$ where $\mathfrak{r} = \overline{\mathfrak{q}}$. Then

$$\overline{\hat{\mathfrak{q}}} = \overline{\Phi_\alpha^{-1}\mathfrak{q}} = \Phi_{\overline{\alpha}}^{-1}\overline{\mathfrak{q}} = \hat{\mathfrak{r}},$$

and so $\hat{p} \in \mathcal{G}_{*,r}^-$. Since $p, \Phi, \hat{p}$ are real it is clear that $\hat{\Phi}$ is also real.

Suppose now that p, Φ satisfy a twisting condition. Then $t_{\alpha,\beta} = t_{\alpha,-\alpha}$ and we have

$$\tau(\Phi_\alpha^{-1}\mathfrak{q}) = \Phi_{-\alpha}^{-1}\tau\mathfrak{q}.$$

$p_{\alpha,E_\alpha^{-1}\mathfrak{q}}$ is therefore well-defined and satisfies the twisting condition. It follows that $\hat{\Phi}$ is also twisted.

■

The complementarity of $\Phi_\alpha^{-1}\mathfrak{q}$ and $\Phi_\beta^{-1}\mathfrak{r}$ is crucial and not guaranteed to hold. However Proposition 1.13 tells us that complementarity is generic and an open condition. Indeed if $\Phi_\alpha^{-1}\mathfrak{q}$ and $\Phi_\beta^{-1}\mathfrak{r}$ are complementary at a point $s \in \Sigma$, then there exists a neighbourhood U of s such that the dressing action is defined on U.

3.7 Bianchi Permutabilty

We conclude the abstract discussion of simple factors with a straightforward proof of the permutability of dressing transforms by simple factors. The proof is seen to rely on nothing more than $\mathcal{G}^+ \cap \mathcal{G}_*^- = \{1\}$ and is a generalisation of [11] Proposition 4.15.

Proposition 3.14
Let $\Phi : \Sigma \to \mathcal{G}^+$ (possibly with reality/twisting conditions) and let $p_i := p_{\alpha_i,\beta_i,\mathfrak{q}_i,\mathfrak{r}_i}$ $i = 1,2$ be simple factors where $\{\alpha_1,\beta_1\} \cap \{\alpha_2,\beta_2\} = \emptyset$. Define parabolic subalgebras

$$\hat{\mathfrak{q}}_1 = p_2(\alpha_1)\mathfrak{q}_1, \qquad \hat{\mathfrak{q}}_2 = p_1(\alpha_2)\mathfrak{q}_2,$$
$$\hat{\mathfrak{r}}_1 = p_2(\beta_1)\mathfrak{r}_1, \qquad \hat{\mathfrak{r}}_2 = p_1(\beta_2)\mathfrak{r}_2,$$

and assume that we are in the generic situation where $(\hat{\mathfrak{q}}_1, \hat{\mathfrak{r}}_1)$, $(\hat{\mathfrak{q}}_2, \hat{\mathfrak{r}}_2)$ are complementary pairs so that we can define simple factors

$$\hat{p}_1 := p_{\alpha_1,\beta_1,\hat{\mathfrak{q}}_1,\hat{\mathfrak{r}}_1}, \qquad \hat{p}_2 := p_{\alpha_2,\beta_2,\hat{\mathfrak{q}}_2,\hat{\mathfrak{r}}_2}.$$

Then $\hat{p}_2 p_1 = \hat{p}_1 p_2$ and so

$$\hat{p}_2 \# (p_1 \# \Phi) = \hat{p}_1 \# (p_2 \# \Phi),$$

supposing they are defined.

Figure 3-1 summarises.

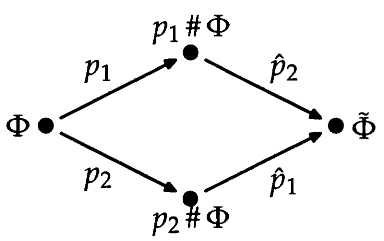

Figure 3-1: Bianchi permutability: dressing by simple factors

Proof $\{\alpha_1, \beta_1\} \cap \{\alpha_2, \beta_2\} = \emptyset \Rightarrow p_2^{-1}$ is holomorphic near α_1, β_1. Apply Corollary 3.13 to $\Phi = p_2^{-1}$ to see that $\hat{p}_1 = p_{\alpha_1, \beta_1, p_2(\alpha_1)q_1, p_2(\beta_1)r_1} \in \mathcal{G}_*^-$ and that $p_1 p_2^{-1} \hat{p}_1^{-1}$ is holomorphic and invertible at α_1, β_1. Similarly, p_1^{-1} is holomorphic near α_2, β_2 and $p_2 p_1^{-1} \hat{p}_2^{-1}$ is holomorphic and invertible at α_2, β_2. Now consider

$$\hat{p}_1 (p_2 p_1^{-1} \hat{p}_2^{-1}) = (p_1 p_2^{-1} \hat{p}_1^{-1})^{-1} \hat{p}_2.$$

The left side of this equation is holomorphic at α_2, β_2 while the right is holomorphic at α_1, α_2. Since simple factors are already holomorphic everywhere else we conclude that the above expression is holomorphic on $\mathbb{P}^1$ which, by Liouville, implies that it is constant. $\lambda = 0$ gives the identity on both sides and hence the result.

■

It is easy to see that Proposition 3.14 restricts to real and/or twisted loops: cf. Proposition 4.5 and [11] Proposition 4.15.

3.8 Coxeter Automorphisms

It has previously been mentioned that one of the reasons for studying loop group dressing is the relation to the transformation theory of solutions to various families of PDEs. This section is motivated by the discussion of the Kuperschmidt–Wilson hierarchy given in [56]. Terng–Uhlenbeck essentially work with maps $\Phi : \Sigma \times \mathbb{C} \to \mathrm{GL}(n, \mathbb{C})$, holomorphic in $\mathbb{C}$ and twisted by a specific order n automorphism τ:

$$\tau\Phi(z) = \Phi(\omega z), \quad \omega = e^{2\pi i/n}.$$

The Φ in question are seen to frame appropriately 'flat' maps into the n-symmetric space $\mathrm{GL}(n)/K$ where the Lie algebra $\mathfrak{k}$ of K is the $+1$-eigenspace of τ in $\mathfrak{gl}(n)$: that is ΦK is a higher order analogue of a curved flat (Definition 1.7). We know from Section 3.3 that the Birkhoff factorisation theorem applies to loops twisted by τ and that $\mathcal{G}_\tau^\pm$ are well defined. Indeed if $g \in \mathcal{G}_\tau^-$ then generically there exist unique $\hat{\Phi} : \Sigma \to \mathcal{G}_\tau^+$ and $\hat{g} : \Sigma \to \mathcal{G}_\tau^-$ such that $g\Phi = \hat{\Phi}\hat{g}$. We will not describe how certain maps Φ correspond to solutions for that would divert us from the similarities of the approach, it suffices to say that the correspondence is seen by considering the logarithmic derivatives of the frames Φ in a similar manner to what will be done in Chapter 4. Instead we will rephrase the dressing theorem of Terng–Uhlenbeck, expressing it more algebraically. We observe that at the Lie algebra level the automorphism τ is in fact a Coxeter automorphism of $\mathfrak{sl}(n)$. We discuss Coxeter automorphisms in the other accessible simple Lie algebras $\mathfrak{so}(2n+1), \mathfrak{so}(2n), \mathfrak{sp}(n)$ and search for the simplest possible maps in $\mathcal{G}_\tau^-$

that one could attempt to dress with: the simple factors. Sadly the theory fails except in $\mathfrak{sl}(n)$, for the dressed $\hat{g} : \Sigma \to \mathcal{G}_\tau^-$ are not simple factors: the distinction comes because simple factors in $\mathfrak{sl}(n)$ will be seen to depend fractionally linearly on $\mathbb{C}$, while in the other algebras there is a fractional quadratic dependence.

Roots and Coxeter automorphisms

Let $\mathfrak{g}$ be a simple complex Lie algebra and let $\mathfrak{h}$ be a Cartan subalgebra (CSA) with root system Δ. For any root $\alpha \in \Delta$, let $\mathfrak{g}^\alpha$ be the root space

$$\mathfrak{g}^\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x,\ \forall h \in \mathfrak{h}\}.$$

Label the simple positive roots $\alpha_1, \ldots, \alpha_n$ and the unique minimal root $\alpha_0 = -\hat{\alpha}$. The order of a root α is its height: the sum of the coefficients of the simple roots in the expansion of α. Define the *Coxeter number* m to be $\operatorname{Ord} \hat{\alpha} + 1$.

Definition 3.15
Since $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha$ *and* $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \subset \mathfrak{g}^{\alpha+\beta}$ *(if* $\alpha + \beta \in \Delta$*) we can define an automorphism: the* Coxeter automorphism *associated to* Δ *is the automorphism* τ *of* $\mathfrak{g}$ *which maps* $x_\alpha \in \mathfrak{g}^\alpha \mapsto \omega^{\operatorname{Ord} \alpha} x_\alpha$ *where* ω *is a primitive* m^{th} *root of unity and* $\tau|_{\mathfrak{h}} = \mathrm{Id}$.

Observe that $\mathfrak{g}^\alpha \mapsto \frac{2\pi i}{m} \operatorname{Ord} \alpha \cdot \mathfrak{g}^\alpha$ is a derivation $\operatorname{ad} \xi$ of $\mathfrak{g}$ and that $\tau = \exp(\operatorname{ad} \xi)$, hence τ is inner. We write $\rho = \exp(\xi)$ so that $\tau = \operatorname{Ad} \rho$ when required.

In section 7 of [45] Kostant discusses the concept of CSAs in apposition:

Definition 3.16
Let $P \in \operatorname{Ad}(\mathfrak{g})$ *be semisimple and* $\mathfrak{g}^P := \{x \in \mathfrak{g} : Px = x\}$. *If* $\mathfrak{g}^P$ *is a CSA then* P *is a* principal element *of* $\operatorname{Ad}(\mathfrak{g})$. *Two CSAs* $\mathfrak{h}_1, \mathfrak{h}_2$ *are in* apposition *with respect to a principal element* $P \in \operatorname{Ad}(\mathfrak{g})$ *iff* $\mathfrak{g}^P = \mathfrak{h}_1$ *and* $\mathfrak{h}_2$ *is P-stable such that* $P|_{\mathfrak{h}_2}$ *has a primitive* m^{th} *root of unity as an eigenvalue.*

In [45, Theorem 7.3] Kostant shows that pairs of CSAs in apposition form a conjugacy class under $\operatorname{Ad}(\mathfrak{g})$, thus once one pair has been found, all others follow. We use the concept of apposition to construct automorphisms of $\mathfrak{g}$.

Proposition 3.17 (Kostant)
Let $X_i \in \mathfrak{g}^{\alpha_i} \setminus \{0\}$ *for* $i = 0, \ldots, n$ *and define* $X := \sum_{i=0}^n X_i$. *The centraliser of* X, $\mathfrak{g}^X := \{x \in \mathfrak{g} : [X, x] = 0\}$ *is another CSA in apposition to* $\mathfrak{h}$.

In Kostant's discussion of the Coxeter automorphism one starts with a CSA $\mathfrak{g}^X$ and simple roots $\beta_1, \ldots, \beta_n$, then defines τ by composition of reflections in $\mathfrak{g}^X$ with respect to the roots. One then works backwards via the apposition definition to see that our definition of Coxeter automorphism is valid.

We now specialise the Coxeter automorphism to specific Lie algebras and use the choice of X to build simple factors. Observe that $\tau X = \omega X$.

The Special linear algebra

$\mathfrak{sl}(n+1)$ has rank n (number of simple roots α_i) and maximal root $\hat{\alpha} = \sum_{j=1}^{n} \alpha_i$. Thus $m = n+1$. The roots are denoted α_{ij}, $1 < j,k < n+1$ where

$$\alpha_{jk} = \begin{cases} \alpha_j & j = k, \\ \alpha_j + \cdots + \alpha_k & j < k, \\ -(\alpha_j + \cdots + \alpha_k) & j > k. \end{cases}$$

Similarly the root spaces $\mathfrak{g}^{\alpha_{jk}}$ are written $\mathfrak{g}_{jk}$.

Let $\mathfrak{h}$ be a CSA with root system Δ, $\tau = \operatorname{Ad}\rho$ the Coxeter automorphism with respect to $(\mathfrak{h}, \Delta)$ and X be chosen as above so that $\mathfrak{g}^X$ is a CSA in apposition to $\mathfrak{h}$. Let v be any eigenvector of X with non-zero eigenvalue e (X is diagonalisable since X is in a CSA) and let ρ move v about $\mathbb{C}^{n+1}$. Now

$$\rho^j X v = \tau^j X \rho^j v = \omega^j X \rho^j v \Longrightarrow X \rho^j v = e\omega^{-j} \rho^j v, \ \forall j = 0, \ldots n. \tag{3.3}$$

Suppose there exists a linear dependence on the set $\{\rho^j v\}_{j=0}^{m-1}$, i.e. $\sum_{j=0}^{m-1} a_j \rho^j v = 0$ for some $a_j \in \mathbb{C}$. Repeatedly apply X to the linear dependence to get the relation

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \cdots & \omega^{1-m} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & \cdots & \omega^{2(1-m)} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-9} & \cdots & \omega^{3(1-m)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{1-m} & \omega^{2(1-m)} & \omega^{3(1-m)} & \cdots & \omega^{(1-m)^2} \end{pmatrix}}_{\underline{\omega}} \begin{pmatrix} a_0 v \\ a_1 \rho v \\ a_2 \rho^2 v \\ a_3 \rho^3 v \\ \vdots \\ a_{m-1} \rho^{m-1} v \end{pmatrix} = 0 \tag{3.4}$$

which clearly holds iff $\det \underline{\omega} = 0$. However observing that $m^{-1}\underline{\omega}^2$ is a permutation matrix, it is easy to see that $\det \underline{\omega} = \pm \lambda m^{m/2}$ where $\lambda = i, 1$ if m is even or odd respectively. Thus $\dim \langle \rho^j v \rangle_{j=0}^{m-1} = m$ and the $\rho^j v$ form a basis of $\mathbb{C}^{n+1}$. In this basis it is clear that ρ has matrix

$$\rho = \begin{pmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix}.$$

This is exactly the automorphism ρ of Terng–Uhlenbeck. Moreover our original CSA $\mathfrak{h}$ is the span of the extended, non-principal, diagonals:

$$\mathfrak{h} = \bigoplus_{j=1}^{n} \left\langle \sum_{i=1}^{n+1} \rho^i v \otimes (\rho^{i+j \text{ (mod } n+1)} v)^* \right\rangle.$$

It is now easy to see that X, the CSA $\mathfrak{g}^X$ and the root spaces $\mathfrak{g}_{jk}$ with respect $\mathfrak{g}^X$ are concisely given in terms of v by

$$X = \sum_{j=0}^{n} e\omega^{-j} \rho^j v \otimes (\rho^j v)^*,$$

$$\mathfrak{g}^X = \bigoplus_{j=0}^{n-1} \left\langle \rho^j v \otimes (\rho^j v)^* - \rho^{j+1} v \otimes (\rho^{j+1} v)^* \right\rangle,$$

$$\mathfrak{g}_{jk} = \left\langle \rho^{j-1} v \otimes (\rho^{k-1} v)^* \right\rangle,\ j,k = 1, \ldots, n+1,$$

where $(\rho^j v)^*$ is the dual basis vector to $\rho^j v$.

We now consider special maps in $\mathcal{G}_\tau^-$ which utilise the root decomposition. Suppose that there exists a map $g : U \subset \mathbb{P}^1 \to \mathrm{Ad}(\mathfrak{sl}(n+1))$ which is twisted in the sense that $\tau g(z) = g(\omega z)$ and can be represented as

$$g(z) = \psi_{\mathfrak{g}^X} \pi_{\mathfrak{g}^X} + \sum_\beta \psi_\beta(z) \pi_\beta,$$

where π_β is projection onto the root space $\mathfrak{g}^\beta$. We already have several conditions on the ψ_β:

- *g twisted*: $\psi_{\tau^j \beta}(z) = \psi_\beta(\omega^j z)$.
- *-ve roots*: $\psi_\beta \psi_{-\beta} = 1$, $\forall \beta$ (which implies that $\psi_{\mathfrak{g}^X} \equiv 1$).
- *g in the group*: if β_1, β_2 are roots such that $\beta_1 + \beta_2$ is also a root, then $\psi_{\beta_1} \psi_{\beta_2} = \psi_{\beta_1 + \beta_2}$.

If we let π_{jk} be projection onto $\mathfrak{g}_{jk}$ then our conditions unpack to read

$$\psi_{j+1,k+1}(z) = \psi_{jk}(\omega^{-1}z), \qquad \psi_{jk}\psi_{kl} = \psi_{jl}, \qquad \psi_{jk}\psi_{kj} = 1.$$

Define $\varphi(z) = \psi_{12}(z)$. It is easy to see that

$$\begin{aligned}
&\psi_{1j}(z) = \varphi(z)\varphi(\omega^{-1}z)\cdots\varphi(\omega^{-(j-2)}z),\\
&\Rightarrow \psi_{jk}(z) = \frac{\varphi(z)\cdots\varphi(\omega^{-(k-2)}z)}{\varphi(z)\cdots\varphi(\omega^{-(j-2)}z)}.
\end{aligned}$$

The construction of g thus comes down to the selection of a single complex-valued function φ satisfying

$$\prod_{j=0}^{n}\varphi(\omega^j z) = 1,$$

because of which we can write the ψ_{jk} more succinctly:

$$g(z) = \pi_{\mathfrak{g}^X} + \sum_{j\neq k}\prod_{l=j-1}^{k-2}\varphi(\omega^{-l}z)\pi_{jk}.$$

It is to be understood that l increases from $j-1$ to $k-2$ (mod m if necessary). The simplest such function is clearly the linear fractional transformation as used by Terng–Uhlenbeck

$$\varphi_\alpha(z) = \frac{z-\omega\alpha}{z-\omega^2\alpha},\ \alpha\in\mathbb{C}. \tag{3.5}$$

The extra factors of ω in φ_α are merely for aesthetics so that $\psi_{jk}(z) = \frac{z-\omega^j\alpha}{z-\omega^k\alpha}$. Note that scaling X by a constant leaves $\mathfrak{g}^X$ unchanged, which explains why Terng–Uhlenbeck choose $\alpha = e$ and define $g_{v,e}$ in terms of the eigenvalue e. We will use the more invariant notation $g_{X,\alpha}$ and refer to $g_{X,\alpha}$ as a *simple factor*.

Dressing by $g_{X,\alpha}$

We now prove an analogue of Theorem 3.12 for the dressing action of $g_{X,\alpha}$. It will be seen that this has exactly the same flavour as Theorem 3.12 in that the dressed loop $\hat{g}$ is also a simple factor.

Given $g_{X,\alpha}$, it is clear that the projection coefficients ψ_{jk} take the form

$$\psi_{jk}(z) = \frac{z-\alpha\omega^j}{z-\alpha\omega^k}.$$

Let $F : \Sigma \to \mathcal{G}_\tau^+$, that is for each $s \in \Sigma$, F is a holomorphic map into $\mathrm{SL}(n+1)$ which is twisted and invertible at all values of $\mathbb{C}$. Write $E = \mathrm{Ad}\, F$.

Theorem 3.18
Where defined, the dressing action of $g_{X,\alpha}(z) = \pi_{\mathfrak{g}^X} + \sum_{j,k} \frac{z-\alpha\omega^j}{z-\alpha\omega^k}\pi_{jk}$ *on* E *is*

$$\hat{E} := g_{X,\alpha} \# E = gE\hat{g}^{-1},$$

where $\hat{g} = g_{\hat{X},\alpha}$ *such that* $\hat{X}$ *is constructed from* $\hat{v} := F^{-1}(\alpha)v$ *in the same fashion as* X *from* v.

The well-definition of the dressing action amounts to the linear independence of the vectors $\rho^j\hat{v} = F^{-1}(\omega^j\alpha)\rho^j v$ and is equivalent to $\hat{X}$ being a CSA. This is an open condition in the manner of Proposition 1.13 since, if $\mathrm{Ord}(\hat{v}) < m$, then $\mathrm{Ord}(\hat{v} + \lambda v) = m$ for small λ.

Proof We argue by showing that when the dressing action is defined, $\hat{g}$ is as claimed. Define $\hat{g} = g_{\hat{X},\alpha}$ and observe that

$$g(z)E(z)\hat{g}^{-1}(z) = \sum_{pqrs} \frac{z-\alpha\omega^p}{z-\alpha\omega^q}\frac{z-\alpha\omega^s}{z-\alpha\omega^r}\pi_{pq}E(z)\hat{\pi}_{rs}. \tag{3.6}$$

We need only worry about the behaviour of $\hat{E}$ at $z = \alpha$ since the behaviour at $\alpha\omega^j$ amounts to a relabelling of indices. Expand E in power series about α with coefficients $E_0 = E(\alpha)$, $E_1 = \left.\frac{\mathrm{d}}{\mathrm{d}z}\right|_{z=\alpha} E(z)$, etc. The coefficients of negative powers of $z - \alpha$ must be zero to ensure the holomorphicity of the result.

- $(z-\alpha)^{-2}$,

$$\sum_{p,s\neq m} \pi_{pm}E_0\hat{\pi}_{ms} = 0.$$

- $(z-\alpha)^{-1}$,

$$\sum_{p,q,s\neq m} (\pi_{\mathfrak{g}^X} + \pi_{pq})E_0\hat{\pi}_{ms} = 0 = \sum_{p,r,s\neq m} \pi_{pm}E_0(\hat{\pi}_{\mathfrak{g}^{\hat{X}}} + \hat{\pi}_{rs}) = \sum_{p,s\neq m} \pi_{pm}E_1\hat{\pi}_{ms}.$$

By the construction of $\hat{X}$ we have $\hat{\mathfrak{g}}_{pq} = F^{-1}(\omega^p\alpha)\mathfrak{g}_{pq}F(\omega^q\alpha)$: this only makes sense in terms of matrix multiplication in $\mathfrak{gl}(n)$. For the first condition, notice that

$$\mathrm{Im}\,\pi_{pm}E_0\hat{\pi}_{ms} = \pi_{pm}E_0\hat{\mathfrak{g}}_{ms} = \pi_{pm}(\mathfrak{g}_{ms}F(\alpha\omega^s)F^{-1}(\alpha)) = 0,$$

since $p \neq m$. For the second

$$\begin{aligned}\operatorname{Im}(\pi_{\mathfrak{g}^X} + \pi_{pq})E_0\hat{\pi}_{ms} &= (\pi_{\mathfrak{g}^X} + \pi_{pq})E_0\hat{\mathfrak{g}}_{ms} = \pi_{mm}(\mathfrak{g}_{ms}F(\alpha\omega^s)F^{-1}(\alpha)) \\ &= \left\langle \pi_{mm}(v \otimes (\rho^s\hat{v})^* F^{-1}(\alpha)) \right\rangle.\end{aligned}$$

However $(\rho^s\hat{v})^*F^{-1}(\alpha)v = (\rho^s\hat{v})^*\hat{v} = 0$ since $s \neq m$, hence the second condition is satisfied. The third condition is very similar to the second. For the E_1 condition notice that the first two conditions show that

$$E_0\hat{\mathfrak{g}}_{ms} \subset \bigoplus_{l\neq m} \mathfrak{g}_{kl} \cap \bigoplus_{k \text{ or } l=m} \mathfrak{g}_{kl} = \bigoplus_{s\neq m} \mathfrak{g}_{ms}.$$

By dimension counting we have $\bigoplus_{s\neq m} \hat{\mathfrak{g}}_{ms} = E_0^{-1} \bigoplus_{s\neq m} \mathfrak{g}_{ms}$. Thus,

$$\operatorname{Im} \pi_{pm}E_1\hat{\pi}_{ms} = \pi_{pm}E_1E_0^{-1}\mathfrak{g}_{ms} = \pi_{pm}[F_1F_0^{-1}, \mathfrak{g}_{ms}] = 0,$$

since $[\mathfrak{sl}(n+1), \mathfrak{g}_{ms}] \subset \bigoplus_{j,k=1}^{m} \mathfrak{g}_{mj} \oplus \mathfrak{g}_{ks}$ and $s \neq m$.

(3.6) is therefore holomorphic at α. Repeating the above calculation at $\alpha\omega^j$ is just a relabelling, while invertibility at α is just the above applied to $\hat{g}E^{-1}g^{-1}$ as in Theorem 3.12. By uniqueness of the dressing action, $\hat{E} = gE\hat{g}$.

■

The Orthogonal algebras

Similarly to the discussion of $\mathfrak{sl}(n+1)$ we exhibit the simplest elements of $\mathcal{G}_\tau^-$ with which one may dress when τ is a Coxeter automorphism and $\mathfrak{g}$ is orthogonal.

Let $\mathfrak{g} = \mathfrak{so}(2n+1)$ with CSA $\mathfrak{h}$ and root system Δ and let $\tau = \operatorname{Ad}\rho$ be the Coxeter automorphism of $\mathfrak{so}(2n+1)$ with respect to Δ. The maximal root is given in terms of the simple roots $\alpha_1, \ldots, \alpha_n$ by

$$\hat{\alpha} = \alpha_1 + 2(\alpha_2 + \cdots + \alpha_n),$$

and so the Coxeter number is $m = 2n$. Let X be chosen with respect to $\mathfrak{h}$ as in Proposition 3.17. We work with the usual representation of $\mathfrak{so}(2n+1)$ on $\mathbb{C}^{2n+1}$. Let $v \in \mathbb{C}^{2n+1}$ be any eigenvector of X with non-zero eigenvalue e. Then, as in (3.3,3.4), we have that the vectors $\rho^j v$, $j = 0, \ldots m-1$ are linearly independent. Consider inner products:

$$\begin{array}{ccc} (X\rho^j v, \rho^k v) & = & -(\rho^j v, X\rho^k v) \\ \| & & \| \\ e\omega^{-j}(\rho^j v, \rho^k v) & = & -e\omega^{-k}(\rho^j v, \rho^k v) \end{array} \implies (\rho^j v, \rho^k v) = 0 \text{ or } k \equiv j + n \mod m.$$

We can therefore choose v such that $(v, \rho^n v) = 1$. Each $\rho^j v$ is thus null and orthogonal to all other $\rho^j v$ except $\rho^{j+n} v$. Finally let $w \in \langle v, \dots, \rho^{m-1} v\rangle^\perp$ be unit length. Note that the eigenvalues of ρ on $\langle w\rangle^\perp$ are ω^j, $j = 0, \dots, m-1$, since

$$\rho \sum_{k=0}^{m-1} \omega^{-jk} \rho^k v = \omega^j \sum_{k=0}^{m-1} \omega^{-jk} \rho^k v.$$

Thus

$$\det \rho|_{\langle w\rangle} = (\det \rho|_{\langle w\rangle^\perp})^{-1} = \exp\left(-\frac{2\pi i}{m} \sum_{j=0}^{m-1} j\right) = -1$$

since m is even. It follows that $\rho w = -w$. Inner product considerations similar to the above tell us that $Xw = 0$. With respect to the isomorphism[9]

$$\bigwedge^2 \mathbb{C}^{2n+1} \cong \mathfrak{so}(2n+1, \mathbb{C}) : u \wedge v \mapsto \big(\lambda \mapsto (u, \lambda)v - (v, \lambda)u\big) \tag{3.7}$$

it is now easy to see that the new Cartan subalgebra $\mathfrak{g}^X$ is given by

$$\mathfrak{g}^X = \left\langle \rho^j v \wedge \rho^{j+n} v \right\rangle_{j=0}^{n-1}, \text{ where } X = \sum_{j=0}^{n-1} e\omega^{-j} \rho^j v \wedge \rho^{j+n} v.$$

It is furthermore clear that the root spaces associated to $\mathfrak{g}^X$ are

$$\mathfrak{g}_{jk} = \left\langle \rho^j v \wedge \rho^k v \right\rangle, \ \mathfrak{g}_j = \left\langle \rho^j v \wedge w \right\rangle, \ j, k = 0, \dots, 2n-1.$$

Now suppose that there exists a map $g : U \subset \mathbb{P}^1 \to \mathrm{Ad}(\mathfrak{so}(2n+1))$ holomorphic near ∞ which is twisted in the sense that $\tau g(z) = g(\omega z)$ and can be written

$$g(z) = \psi_{\mathfrak{g}^X} \pi_{\mathfrak{g}^X} + \sum_\beta \psi_\beta(z) \pi_\beta,$$

where π_β is projection onto the root space $\mathfrak{g}^\beta$. Similarly to the discussion of $\mathfrak{sl}(n+1)$ we write $\psi_{jk} := \psi_{\rho^j v \wedge \rho^k v}$ and $\varphi_j := \psi_{\rho^j v \wedge w}$ for ψ_β and see that the twisting and group-valued conditions lead to the following:

$$\begin{aligned}
&\psi_{j+1,k+1}(z) = \psi_{jk}(\omega^{-1} z), \qquad && \varphi_{j+1}(z) = \varphi_j(\omega^{-1} z), \\
&\psi_{jk} \psi_{j+n,k+n} = 1, && \varphi_j \varphi_{j+n} = 1, \\
&\psi_{jk} \psi_{k+n,q} = \psi_{jq}, && \varphi_j \psi_{j+n,p} = \varphi_p.
\end{aligned}$$

[9] Valid for any dimension, not just odd, and for the real orthogonal algebras.

Note that $\psi_{jk} = \psi_{kj}$ so we have a lot more conditions than just those written above. We can reduce these conditions further by realising that only $\varphi := \varphi_0$ is important, for the above conditions give

$$\varphi_j(z) = \varphi(\omega^{-j}z), \qquad \psi_{jk}(z) = \varphi(\omega^{-j}z)\varphi(\omega^{-k}z).$$

The only remaining restriction is

$$\varphi(z)\varphi(-z) = 1.$$

Recall that g is required to be holomorphic near ∞ thus forcing the same property in φ. The simplest choice is to let $\varphi(z) = \frac{z-\alpha}{z+\alpha}$ for some fixed $\alpha \in \mathbb{C}^\times$. In such a case $g(z)$ is quadratic fractional with poles at $\omega^j\alpha$, $j = 0, \ldots, 2n-1$.

The $\mathfrak{so}(2n)$ case is marginally more complicated than the above. Build X in the same way as for $\mathfrak{so}(2n+1)$: this time the Coxeter automorphism $\tau = \operatorname{Ad}\rho$ has order $m = 2(n-1)$. We can find an eigenvector v of X such that $\{\rho^j v\}_{j=0}^{2n-3}$ are null vectors with $(v, \rho^{(n-1)}v) = 1$ and $(v, \rho^j v) = 0$ if $j \neq n-1$. $W := \langle \oplus_j \rho^j v\rangle^\perp$ is 2-dimensional and, by a similar calculation to above, ρ has determinant -1 when restricted to W. Indeed we may choose w isotropic in W and then it is clear that $\rho w \neq w$ and $\rho^2 w = w$. Thus $W = \langle w, \rho w\rangle$. Furthermore XW is easily seen to be zero by taking repeated inner products. The CSA $\mathfrak{g}^X$ is now

$$\mathfrak{g}^X = \left\langle \rho^j v \wedge \rho^{j+n-1}v,\ w \wedge \rho w \right\rangle_{j=0}^{n-2} \quad \text{where} \quad X = \sum_{j=0}^{n-2} e\omega^{-j}\rho^j v \wedge \rho^{j+n-1}v.$$

We are looking for $g \in \mathcal{G}_\tau^-$ of the form

$$g(z) = \pi_{\mathfrak{g}^X} + \sum_{j=0}^{n} \left(\varphi_j(z)\pi_j + \hat{\varphi}_j(z)\hat{\pi}_j\right) + \sum_{j<k\neq j+n-1} \psi_{jk}(z)\pi_{jk}$$

where $\pi_{\mathfrak{g}^X}, \pi_j, \hat{\pi}_j, \psi_{jk}$ are projections onto $\mathfrak{g}^X, \langle \rho^j v \wedge w\rangle, \langle \rho^j v \wedge \rho w\rangle, \langle \rho^j v \wedge \rho^k v\rangle$ respectively. The twisting and group-valued conditions yield the following:

$$\begin{aligned} &\varphi_j \hat{\varphi}_{j+n-1} = 1, && \psi_{j+n-1,k} = \varphi_k/\varphi_j = \hat{\varphi}_j/\hat{\varphi}_k, \\ &\psi_{jk}\psi_{j+n-1,k} = \psi_{kp}, && \varphi_j(z) = \hat{\varphi}_{j-1}(\omega^{-1}z) \end{aligned}$$

The solution differs slightly depending on the parity of n, however the working is very similar to the $\mathfrak{so}(2n+1)$ case so we will omit most of it. In both cases we will write $\varphi = \varphi_0, \hat{\varphi} = \hat{\varphi}_0$ for brevity. If n is even the above conditions give

$$\varphi(z)\varphi(-z) = 1 = \hat{\varphi}(z)\hat{\varphi}(-z),$$
$$\varphi(z)\varphi(\omega^j z)\hat{\varphi}(-z)\hat{\varphi}(-\omega^j z) = 1, \ \forall j = 1, \dots n-2.$$

We therefore have a choice of one function φ satisfying $\varphi(z)\varphi(-z) = 1$, with $\hat{\varphi} := \pm\varphi$ and all ψ_{jk} being products of these. The simplest φ are once again fractional linear and so the simplest possible g are quadratic fractional.

When n is odd we instead have

$$\varphi(z)\hat{\varphi}(-z) = 1,$$
$$\varphi(z)\varphi(\omega^j z)\varphi(-z)\varphi(-\omega^j z) = 1, \ \forall j = 1, \dots n-2.$$

For $n \geq 5$ we have a choice of a single function φ satisfying $\varphi(z)\varphi(-z) = \pm 1$ and again can do no better than to have g quadratic fractional. Exceptionally when $n = 3$ we have only one condition

$$\varphi(z)\varphi(iz)\varphi(-z)\varphi(-iz) = 1.$$

If we choose $\varphi(z) = \frac{z-\alpha}{z-i\alpha}$ it can easily be seen that all coefficients of g are fractional *linear*. This reflects the exceptional isomorphism $\mathfrak{sl}(4) \cong \mathfrak{so}(6)$ (cf. Klein correspondence (2.1)).

There is no reason to expect the dressing action of quadratic fractional g to be straightforward to calculate in the manner of fractional linear g. Suppose that $gE = \hat{E}\hat{g}$ is the Birkhoff decomposition for a quadratic fractional g and that $\hat{g}$ is also a simple factor. For the 'usual' proof of the calculation of the dressing action to hold we must be able to build a $\hat{v}$ ($= E^{-1}(\alpha)v$ say) for which $\{\rho^j\hat{v}\}_{j=0}^{2n}$ satisfy the independent orthogonality conditions $(\hat{v}, \rho^j\hat{v}) = 0$, $j = 0, \dots, n-1$, $(\hat{v}, \rho^n\hat{v}) = 1$. The set of such $\hat{v}$ comprise a codimension $n+1$ subset of $\mathbb{R}^{2n+1}$ and so it is extremely easy to fall outside the set of allowable $\hat{v}$.

One may similarly investigate the simple factors for $\mathfrak{sp}(n)$ via the isomorphism

$$S^2\mathbb{C}^{2n} \cong \mathfrak{sp}(n, \mathbb{C}) \quad \text{via} \quad (u \odot v)w = \omega(u,w)v + \omega(v,w)u \tag{3.8}$$

where ω is the symplectic form on $\mathbb{C}^{2n}$ with respect to which $\mathfrak{sp}(n)$ is defined. It is not difficult to see that one obtains quadratic fractional simple factors in exactly the same fashion as for $\mathfrak{so}(2n)$.

Chapter 4

Bäcklund-type Transforms of $\mathfrak{p}$-flat Maps

4.1 Introduction

The various applications of loop group dressing arguments in the literature centre on the realisation that certain geometric objects correspond to a subset of maps into a group of positive loops $\mathcal{G}^+$. In the happy situation where the local dressing action (Definition 3.5) of the negative loops $\mathcal{G}^-$ preserves this subset, we have transformations of these geometric objects. These could be transformations of harmonic maps [60], CMC surfaces [26], hierarchies of PDEs [56] or, as is the subject of this chapter, Burstall's $\mathfrak{p}$-flat maps [11]. A $\mathfrak{p}$-flat map may be viewed as a local integral of the Maurer-Cartan form of a curved flat in a symmetric space G/K. The 1-parameter integrability of such maps (related to the spectral deformation of the curved flat) allows us to build a correspondence of $\mathfrak{p}$-flat maps with certain maps $\Phi : \Sigma \to \mathcal{G}^+_{*,r,\tau}$ into the loop group of based, real, twisted, positive loops. In dressing Φ pointwise by (based) negative loops it is observed that the set of maps into $\mathcal{G}^+_{*,r,\tau}$ corresponding to $\mathfrak{p}$-flat maps is preserved and so we get a local dressing action of $\mathcal{G}^-_{*,r,\tau}$ on $\mathfrak{p}$-flat maps. When the Lie group G is such that the negative loop group contains simple factors, we may apply the theory of Chapter 3 and calculate the resulting transforms. For concreteness we explicitly calculate all the simple factors and their dressing action for $\mathfrak{p}$-flat maps related to symmetric $\mathrm{O}(p,q)$-spaces. We conclude with a discussion of the *Bäcklund transform for O-surfaces* of Schief–Konopelchenko [53], which describes a transform of several classical families of surfaces in $\mathbb{R}^n$. We demonstrate that this transform, although phrased in completely different language, corresponds exactly to the dressing of a subset of the $\mathfrak{p}$-flat maps into orthogonal algebras by certain simple factors.

4.2 $\mathfrak{p}$-flat Maps and their Flat Frames

Recall the discussion of symmetric spaces and curved flats (Definitions 1.5 and 1.7). Let G/K be a symmetric space (in practice we will assume G is semisimple and centre-free as on page 58) with d-parallel[1] symmetric involution τ whose $\pm$-eigenspace decomposition is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

We have seen in Section 1.1 that the tangent space to G/K at a point gK can be identified with $\mathrm{Ad}(g)\mathfrak{p}$ via the isomorphism

$$\mathrm{Ad}\, g\mathfrak{p} \cong T_{gK}G/K : \xi \mapsto \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(t\xi) \cdot gK. \tag{4.1}$$

We therefore have an identification of $T(G/K)$ with a subbundle of the trivial bundle $G/K \times \mathfrak{g}$.

Recall that a *curved flat* is an immersion $\sigma : \Sigma \to G/K$ such that each $\mathrm{d}\sigma(T_s\Sigma)$ is an Abelian subalgebra of $\mathfrak{g}$ under the above identification. Curved flats are only of secondary importance to this chapter, far more important are the associated $\mathfrak{p}$-flat maps of Burstall [11]. A curved flat σ is *framed* by maps $\Phi : \Sigma \to G$ (i.e. σ is the quotient space ΦK) defined up to the right action of maps $\Sigma \to K$. The derivative of σ under the soldering identification (4.1) has the form

$$\mathrm{d}\sigma(X_s) = \Big(\sigma(s), \mathrm{Ad}\,\Phi(\mathrm{Proj}_{\mathfrak{p}}(\Phi^{-1}\mathrm{d}\Phi(X_s)))\Big) = (\sigma(s), \mathcal{N}(X_s))$$

for any frame Φ and where $\mathcal{N}$ is the Maurer–Cartan form of G/K along σ. Let $A_{\mathfrak{k}}, A_{\mathfrak{p}}$ be the $\mathfrak{k}, \mathfrak{p}$ components of $\Phi^{-1}\mathrm{d}\Phi$ for some frame Φ. As observed in [11], $A_{\mathfrak{k}}$ satisfies the Maurer-Cartan equations

$$\mathrm{d}A_{\mathfrak{k}} + \frac{1}{2}[A_{\mathfrak{k}} \wedge A_{\mathfrak{k}}] = 0,$$

and can therefore be locally integrated to give a map $k : \Sigma \to K$ such that $k^{-1}\mathrm{d}k = A_{\mathfrak{k}}$. The frame $\hat{\Phi} := \Phi k^{-1}$ now satisfies

$$\hat{\Phi}^{-1}\mathrm{d}\hat{\Phi} = \mathrm{Ad}(k)A_{\mathfrak{p}},$$

which takes values in $\mathfrak{p}$ since $\mathfrak{p}$ is a reductive factor. Burstall defines a *flat frame* of a curved flat σ to be a frame Φ whose logarithmic derivative is $\mathfrak{p}$-valued. Any other flat frame of σ is Φk where $k : \Sigma \to K$ is constant.

Flat frames give rise to the maps into $\mathfrak{p}$ with which this chapter is predominantly

[1] I.e. we follow Definition 1.5 part 1.

concerned. Let Φ be a flat frame and let $A_\mathfrak{p} = \Phi^{-1}\mathrm{d}\Phi$. Then $A_\mathfrak{p}$ satisfies the Maurer–Cartan equations

$$\mathrm{d}A_\mathfrak{p} + \frac{1}{2}[A_\mathfrak{p} \wedge A_\mathfrak{p}] = 0$$

which, since $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$, splits to give $\mathrm{d}A_\mathfrak{p} = 0 = [A_\mathfrak{p} \wedge A_\mathfrak{p}]$. We can therefore locally integrate $A_\mathfrak{p}$ to get a map $\psi : \Sigma \to \mathfrak{p}$.

Definition 4.1
Let $\psi : \Sigma \to \mathfrak{p}$ be an immersion of a manifold Σ. ψ is a $\mathfrak{p}$-flat map *if the image of each tangent space $T_s\Sigma$, $s \in \Sigma$ is an Abelian subalgebra of $\mathfrak{p}$. Expressed algebraically*

$$\psi \text{ is } \mathfrak{p}\text{-flat} \iff [\mathrm{d}\psi \wedge \mathrm{d}\psi] = 0.$$

Remark: $\mathfrak{p}$-flat maps were invented by Burstall [11] to aid the study of isothermic surfaces in S^n, indeed $\mathfrak{p}$-flat maps into $\mathbb{R}^n \wedge \mathbb{R}^{1,1}$ correspond to Christoffel pairs of isothermic surfaces in S^n. We shall generalise this idea in Chapter 5.

Observe that if ψ is a $\mathfrak{p}$-flat map then $\psi + \text{const}$ is also $\mathfrak{p}$-flat since the only conditions on ψ are differential. Also note that $A_\mathfrak{p} = \Phi^{-1} \cdot \mathcal{N}$ so that curved flats give rise to $\mathfrak{p}$-flat maps via the integration of the (conjugated) Maurer–Cartan form: this will be important in Section 5.8.

While curved flats give rise to $\mathfrak{p}$-flat maps through a choice of flat frame, the converse is just as fruitful, for $\mathfrak{p}$-flat maps exhibit a 1-parameter integrability. Since $\mathrm{d}(\mathrm{d}\psi)$ is trivially zero, $z\mathrm{d}\psi$ satisfies a Maurer–Cartan equation for each $z \in \mathbb{C}$:

$$\mathrm{d}(z\mathrm{d}\psi) + \frac{1}{2}[z\mathrm{d}\psi \wedge z\mathrm{d}\psi] = 0.$$

If we denote by $G^\mathbb{C}$ the complexification of G,[2] then each $z\mathrm{d}\psi$ can be locally integrated to give a family of maps $\Phi_z : \Sigma \to G^\mathbb{C}$ satisfying

$$\Phi_z^{-1}\mathrm{d}\Phi_z = z\mathrm{d}\psi. \tag{4.2}$$

Such Φ_z exist locally up to left multiplication by a constant. From now on we restrict ourselves to $\mathfrak{p}$-flat maps from simply connected manifolds Σ with a fixed base point o. Simple-connectedness means that each Φ_z is globally defined. Insisting on the initial condition $\Phi_z(o) = \mathrm{Id}$ fixes each Φ_z uniquely. Note immediately that $\Phi_0 \equiv 1$.

[2]Assume that $G^\mathbb{C}$ is well-defined. For semisimple $\mathfrak{g}$ we can define suitable centre-free $G = \mathrm{Int}(\mathfrak{g})$, $G^\mathbb{C} = \mathrm{Int}(\mathfrak{g}^\mathbb{C})$, as discussed on page 58.

Definition 4.2
$\Phi : \Sigma \to \mathrm{Map}(\mathbb{C}, G^{\mathbb{C}})$ *(defined pointwise by* $\Phi(z) := \Phi_z$*) is a* based extended flat frame *of* ψ.

The original $\mathfrak{p}$-flat map can be recovered from Φ up to a base value via the *Sym formula* ([11] prop 4.2):

$$\psi = \left.\frac{\partial \Phi}{\partial z}\right|_{z=0} + \psi(o). \tag{4.3}$$

As previously remarked, if ψ is $\mathfrak{p}$-flat, then adding a constant gives another $\mathfrak{p}$-flat map. From the definition of the based extended flat frame it is clear that $\psi + \mathrm{const}$ has the *same* Φ as ψ. The Sym formula gives a bijection between $\mathfrak{p}$-flat maps and pairs of based extended flat frames and base values: $\psi \leftrightarrow (\Phi, \psi(o))$.

The based extended frame of a $\mathfrak{p}$-flat map frames a 1-parameter family of curved flats: $\sigma_t := \Phi_t K$ is a curved flat for each $t \in \mathbb{R}^\times$. The family σ_t is the *associated family* or *spectral deformation* of $\sigma = \Phi_1 K$ and is defined up to the left action of G. This family will play a starring role in the discussion of Darboux transforms of isothermic submanifolds in Chapter 5.

4.3 Loop Groups and the Dressing Action

Observe that the conditions defining Φ_z (4.2) and $\Phi_z(o) = \mathrm{Id}$) are holomorphic in z and so Φ depends holomorphically on z. Furthermore the fact that $d\psi$ is $\mathfrak{p}$-valued tells us that Φ is both twisted ($\tau\Phi_z = \Phi_{-z}$) and real ($\overline{\Phi_z} = \Phi_{\bar{z}}$) in the sense of Section 3.3. The based extended frame Φ therefore takes values in the loop group

$$\mathcal{G}^+ := \{g : \mathbb{C} \to G^{\mathbb{C}} \text{ holom} : g(0) = \mathrm{Id},\ \tau g(z) = g(-z),\ \overline{g(z)} = g(\bar{z})\},$$

i.e. $\mathcal{G}^+ = \mathcal{G}^+_{*,r,\tau}$.

We can now apply the dressing theorems of Chapter 3 to the extended flat frames Φ, for we have shown that the Birkhoff factorisation theorem provides a local action # of the loop group

$$\mathcal{G}^- := \{g : \mathrm{dom}(g) \to G^{\mathbb{C}} \text{ holom } : g(0) = \mathrm{Id},\ \tau g(z) = g(-z),\ \overline{g(z)} = g(\bar{z})\}$$

on $\mathcal{G}^+$, where U is $\mathbb{P}^1$ minus a finite number of points in $\mathbb{C}$. The important fact for the application of dressing to $\mathfrak{p}$-flat maps is that the dressed map $\hat{\Phi} = g_-\#\Phi$ of an extended flat frame is the extended flat frame of a new $\mathfrak{p}$-flat map. An extended flat frame is a solution to (4.2) with $\Phi_0 \equiv \mathrm{Id}$ but without the imposition of an initial condition $\Phi_z(o) = \mathrm{Id}$. This is clarified by the following lemma.

Lemma 4.3
$\Phi : \Sigma \to \mathcal{G}^+$ is an extended flat frame iff $\Phi^{-1}\mathrm{d}\Phi$ has a simple pole at ∞.

The proof (Lemma 4.1 in [11]) relies on the interaction of the power series expansion of $\Phi^{-1}\mathrm{d}\Phi$ with the reality and twisting conditions. It is worth stressing that we are viewing Φ as a map $\Sigma \to \mathcal{G}^+$ and *not* as a map $\Sigma \times \mathbb{C} \to G^{\mathbb{C}}$ as is done when we apply the Sym formula (4.3).

Proposition 4.4
If Φ is a based extended flat frame, then $\hat{\Phi} := g_- \# \Phi$ defined pointwise by

$$(g_- \# \Phi)(p) := g_- \# \Phi(p)$$

is the based extended flat frame of a new $\mathfrak{p}$-flat map

$$\hat{\psi} = g_- \# \psi = \psi + \left.\frac{\partial}{\partial z}\right|_{z=0} \hat{g}_-^{-1}$$

giving a local action of $\mathcal{G}^-$ on $\mathfrak{p}$-flat maps.

Proof Consider the logarithmic derivative of $\hat{\Phi}$:

$$\hat{\Phi}^{-1}\mathrm{d}\hat{\Phi} = \mathrm{Ad}\,\hat{g}_-(\Phi^{-1}\mathrm{d}\Phi - \hat{g}_-^{-1}\mathrm{d}\hat{g}_-). \tag{4.4}$$

Since $\hat{g}_-$ is holomorphic near ∞, $\hat{\Phi}^{-1}\mathrm{d}\hat{\Phi}$ has the same pole as $\Phi^{-1}\mathrm{d}\Phi$ and so, by Lemma 4.3 is an extended flat frame. For the base point:

$$\hat{\Phi}_z(o) = g_-(z)\Phi_z(o)\hat{g}_-^{-1}(o,z) = g_-(z)\hat{g}_-^{-1}(o,z) \in \mathcal{G}^+ \cap \mathcal{G}^- = \{\mathrm{Id}\}.$$

Apply the Sym formula to get a local action on $\mathfrak{p}$-flat maps

$$\begin{aligned}\hat{\psi} = g_- \# \psi &:= \left.\frac{\partial}{\partial z}\right|_{z=0} (g_- \# \Phi) + \psi(o) - \left.\frac{\partial}{\partial z}\right|_{z=0} g_- \\ &= \psi + \left.\frac{\partial}{\partial z}\right|_{z=0} \hat{g}_-^{-1}.\end{aligned} \tag{4.5}$$

This is an action because

$$\begin{aligned}g_2 \#(g_1 \# \psi) &= g_2 \# \left(\left.\frac{\partial}{\partial z}\right|_{z=0} (g_1 \# \Phi) + \psi(o) - \left.\frac{\partial}{\partial z}\right|_{z=0} g_1 \right) \\ &= \left.\frac{\partial}{\partial z}\right|_{z=0} (g_2 \#(g_1 \# \Phi)) + \psi(o) - \left.\frac{\partial}{\partial z}\right|_{z=0} (g_1 + g_2)\end{aligned}$$

$$= \left.\frac{\partial}{\partial z}\right|_{z=0} ((g_2 g_1)\#\Phi) + \psi(o) - \left.\frac{\partial}{\partial z}\right|_{z=0} (g_2 g_1)$$
$$= (g_2 g_1)\#\psi.$$

■

Care is required if the second line of (4.5) is taken as a definition. It seems to suggest that the local action on $\mathfrak{p}$-flat maps is commutative, but this is not the case: by considering the action on frames, it is clear that $\hat{g}_1, \hat{g}_2$ depend on the *order* of the action of g_1, g_2, i.e. $\widehat{g_1 g_2} \neq \widehat{g_2 g_1}$.

Equation 4.4 contains more information, for,

$$\mathrm{d}\hat{\psi} = \mathrm{Ad}\,\hat{g}_-(\mathrm{d}\psi - z^{-1}\hat{g}_-^{-1}\mathrm{d}\hat{g}_-)\mathrm{d}\psi \Rightarrow \mathrm{d}\hat{\psi} = \mathrm{Ad}\,\hat{g}_-(\infty)\mathrm{d}\psi. \tag{4.6}$$

The derivatives of $\psi, \hat{\psi}$ have the same rank, Jordan form, etc. This is important for various applications of the dressing.

4.4 Dressing $\mathfrak{p}$-flat Maps by Simple Factors

Now that we have a dressing action (4.5) of $\mathcal{G}^-$ on $\mathfrak{p}$-flat maps, we can specialise to the action of the simple factors of Chapter 3 which should be algebraically computable. Recall (Section 3.5) that the complexification of $\mathfrak{g}^{\mathbb{C}}$ must contain one of the algebras $\mathfrak{sl}, \mathfrak{so}, \mathfrak{sp}, \mathfrak{e}_6, \mathfrak{e}_7$ as a simple ideal if we are to have any hope of simple factors existing. The simple factors in $\mathcal{G}^-$ take the final, real & twisted, form in the table of Theorem 3.11. Let $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{g}^{\mathbb{C}}$ such that $\tau\mathfrak{q}$ is complementary to $\mathfrak{q}$. Furthermore let $\alpha \in \mathbb{R}^\times \cup i\mathbb{R}^\times$ and

$$\begin{cases} \overline{\mathfrak{q}} = \mathfrak{q} & \text{if } \alpha \in \mathbb{R}^\times, \\ \overline{\mathfrak{q}} = \tau\mathfrak{q} & \text{if } \alpha \in i\mathbb{R}^\times. \end{cases}$$

Applying Corollary 3.13 we see that the dressing action of $p_{\alpha,\mathfrak{q}}$ on the extended flat frame Φ is defined iff $(\Phi_\alpha^{-1}\mathfrak{q}, \Phi_{-\alpha}^{-1}\tau\mathfrak{q})$ are complementary, in which case we have

$$p_{\alpha,\mathfrak{q}}\#\Phi = \hat{\Phi} = p_{\alpha,\mathfrak{q}}\Phi p^{-1}_{\alpha,\Phi_\alpha^{-1}\mathfrak{q}}.$$

Meanwhile (4.5) tells us that the action of $p_{\alpha,\mathfrak{q}}$ on the $\mathfrak{p}$-flat map ψ is

$$\begin{aligned} p_{\alpha,\mathfrak{q}} \# \psi = \hat{\psi} &= \psi + \frac{\partial}{\partial z}\bigg|_{z=0} p^{-1}_{\alpha,\Phi_\alpha^{-1}\mathfrak{q}} \\ &= \psi + \frac{\partial}{\partial z}\bigg|_{z=0} \exp\left(-\ln \frac{\alpha - z}{\alpha + z}\hat{\xi}\right) \\ &= \psi + \frac{2}{\alpha}\hat{\xi} \end{aligned} \tag{4.7}$$

where $\hat{\xi}$ is the canonical element of $(\Phi_\alpha^{-1}\mathfrak{q}, \Phi_{-\alpha}^{-1}\tau\mathfrak{q})$.

This result is more powerful than may at first be apparent, for it tells us that once the based extended flat frame of a $\mathfrak{p}$-flat map ψ is known (finding this explicitly usually involves a difficult integration) then we can algebraically calculate the based extended flat frames of any transform of ψ by simple factors. One may therefore start with a seed $\mathfrak{p}$-flat map for which an explicit framing is known and easily calculate its transforms.

We have seen that in order to calculate the dressing action of a simple factor $p_{\alpha,\mathfrak{q}}$ on a $\mathfrak{p}$-flat map we need only find the canonical element $\hat{\xi}$ of the dressed parabolic subalgebras. While the new parabolic subalgebras are easy to calculate, the action on canonical elements is highly non-trivial as the following summary of the steps required in order to calculate $\hat{\xi}$ shows. Given a simple factor $p_{\alpha,\mathfrak{q}}$ (and thus a grading of the Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{q}^\perp \oplus (\mathfrak{q} \cap \tau\mathfrak{q}) \oplus \tau\mathfrak{q}^\perp$) and a $\mathfrak{p}$-flat map ψ:

- Integrate $\Phi_z^{-1}\mathrm{d}\Phi_z = z\mathrm{d}\psi$ to find the based extended flat frame of ψ and define $\hat{\mathfrak{q}} := \Phi_\alpha^{-1}\mathfrak{q}$;
- Make sure that $\hat{\mathfrak{q}}^\perp \cap \tau\hat{\mathfrak{q}} = \{0\}$ so that $p_{\alpha,\mathfrak{q}} \# \psi$ is defined;
- $\hat{\mathfrak{q}}, \tau\hat{\mathfrak{q}}$ are complementary so define $\hat{\xi}$ by setting

$$\mathrm{ad}(\hat{\xi}) = \begin{cases} 1 & \text{on } \hat{\mathfrak{q}}^\perp, \\ 0 & \text{on } \hat{\mathfrak{q}} \cap \tau\hat{\mathfrak{q}}, \\ -1 & \text{on } \tau\hat{\mathfrak{q}}^\perp. \end{cases}$$

Once the based extended flat frame is known, every step of the above process is algebraic.

We conclude the abstract discussion of simple factors with a rephrasing of the Bianchi permutability theorem of Section 3.7: this is exactly Proposition 4.15 of [11].

Proposition 4.5
Given a $\mathfrak{p}$*-flat map* ψ *and two dressings,* $p_{\alpha,\mathfrak{q}_1} \# \psi$*,* $p_{\beta,\mathfrak{q}_2} \# \psi$ *such that* $\alpha^2 \neq \beta^2$*,*[3] *define*

$$\mathfrak{q}_1' = \operatorname{Ad} p_{\beta,\mathfrak{q}_2}(\alpha)\mathfrak{q}_1,$$
$$\mathfrak{q}_2' = \operatorname{Ad} p_{\alpha,\mathfrak{q}_1}(\beta)\mathfrak{q}_2.$$

Then $p_{\alpha,\mathfrak{q}_1'} p_{\beta,\mathfrak{q}_2} = p_{\beta,\mathfrak{q}_2'} p_{\alpha,\mathfrak{q}_1}$ *and so*

$$p_{\alpha,\mathfrak{q}_1'} \#(p_{\beta,\mathfrak{q}_2} \# \psi) = p_{\beta,\mathfrak{q}_2'} \#(p_{\alpha,\mathfrak{q}_1} \# \psi).$$

4.5 Examples of Simple Factor Dressing

In this section we make contact with Burstall's description [11] of the simple factors for $G = \mathrm{SO}(n+1,1)$ with which one may dress $\mathfrak{p}$-flat maps into $\mathbb{R}^n \wedge \mathbb{R}^{1,1} \subset \mathfrak{so}(n+1,1)$ (via the usual isomorphism (3.7)). We instead find all simple factors for G the real form $\mathrm{SO}(i,j)$, $i+j=d$ of $\mathrm{SO}(d,\mathbb{C})$. In a similar manner to the discussion of simple factors twisted by Coxeter automorphisms (Section 3.8) we consider the usual representation of the complexified orthogonal group on $\mathbb{C}^n$. For concreteness we will work with $G^{\mathbb{C}} = \mathrm{O}(d,\mathbb{C})$ which has centre $\pm\mathrm{Id}$ rather than $\mathrm{SO}(d,\mathbb{C})$ which has centre dependent on the parity of d. We start by finding all simple factors regardless of reality and twisting conditions. Given a simple factor $g(z) = \gamma\left(\frac{1-\alpha^{-1}z}{1-\beta^{-1}z}\right)$, the crucial equation (3.1) unpacks to give us more information about the canonical element ξ:

$$\gamma(e^{2\pi i}) = \exp(2\pi i\xi) = \pm\mathrm{Id}.$$

The Jordan decomposition theorem says that ξ is diagonalizable on $\mathbb{C}^d$ with *half-integer* eigenvalues. Suppose $\mathbb{C}^d = \bigoplus_j V_j$ where ξ has eigenvalue e_j on V_j. The isomorphism (3.7) tells us that $\operatorname{ad}\xi$ has eigenvalue $e_j + e_k$ on $V_j \wedge V_k$ and so $e_j + e_k \in \{0, \pm 1\}$, $\forall j,k$. It is clear that the only possibilities are to have $(e_0, e_\pm) = (0, \pm 1)$ or $e_\pm = \pm 1/2$.

In the first case $V_\pm$ are lines (since $\bigwedge^2 V_\pm$ must vanish) and $V_0 = (V_+ \oplus V_-)^\perp$. It follows that the graded decomposition of $\mathfrak{so}(d)$ is

$$\mathfrak{so}(d) \cong \bigwedge^2 \mathbb{C}^d = \underbrace{V_+ \wedge V_0}_{\mathfrak{g}_+} \oplus \underbrace{V_+ \wedge V_- \oplus \bigwedge^2 V_0}_{\mathfrak{g}_0} \oplus \underbrace{V_- \wedge V_0}_{\mathfrak{g}_-}.$$

Furthermore it is easy to see from $[V_+ \wedge V_-, V_+ \wedge V_0] \subset V_+ \wedge V_0$ that $V_\pm$ are isotropic

[3] This not only precludes $\alpha = \beta$, but stops us encountering the duality $p_{\alpha,\mathfrak{q}} = p_{-\alpha,\tau\mathfrak{q}}$.

lines. It follows from Theorem 3.11 that the simple factors take the form

$$p_{\alpha,\beta}(z) = \frac{1-\alpha^{-1}z}{1-\beta^{-1}z}\pi_+ + \pi_0 + \frac{1-\beta^{-1}z}{1-\alpha^{-1}z}\pi_-, \quad \alpha,\beta \in \mathbb{C}^\times$$

where π_j is projection in $\mathbb{C}^d$ onto V_j.

In the second case $\bigwedge^2 V_\pm$ must be Abelian and so $V_\pm$ are isotropic. Decompositions $\mathbb{C}^d = V_+ \oplus V_-$ into isotropic spaces can only happen if d is even and $\dim V_\pm = d/2$. $\mathfrak{so}(d)$ decomposes as

$$\mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_- = \bigwedge^2 V_+ \oplus V_+ \wedge V_- \oplus \bigwedge^2 V_-,$$

from which we recover the simple factors

$$p_{\alpha,\beta}(z) = \sqrt{\frac{1-\alpha^{-1}z}{1-\beta^{-1}z}}\pi_+ + \sqrt{\frac{1-\beta^{-1}z}{1-\alpha^{-1}z}}\pi_-, \quad \alpha,\beta \in \mathbb{C}^\times.$$

Suppose now that we are dressing $\mathfrak{p}$-flat maps, so that there exist reality and twisting conditions. To follow Burstall, we let $\mathbb{R}^d$ be a non-degenerate real subspace of $\mathbb{C}^d$ and let $\mathbb{R}^d = \mathbb{R}^m \oplus \mathbb{R}^n$ be an orthogonal decomposition ($\mathbb{R}^m, \mathbb{R}^n$ may have indefinite signature). Define $\rho := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ with respect to this decomposition so that $\tau := \operatorname{Ad}\rho \in \operatorname{Aut}\mathfrak{g}$ is our symmetric involution. It is clear that $\mathfrak{p} \cong \mathbb{R}^m \wedge \mathbb{R}^n$. From our expressions for the simple factors, or indeed appealing once again to Theorem 3.11, we see that supposing p to be twisted with respect to τ forces $V_- = \tau V_+$ and $\beta = -\alpha$ in both cases. Imposing the reality condition with respect to the real form $\mathrm{O}(\mathbb{R}^d)$ we see that either $\overline{V_+} = V_+$ and $\alpha \in \mathbb{R}^\times$ or $\overline{V_+} = V_-$ and $\alpha \in i\mathbb{R}^\times$. To summarise, there are two possibilities:

1. Choose a scalar $\alpha \in \mathbb{R}^\times \cup i\mathbb{R}^\times$ and a null line L where,
   - $\alpha \in \mathbb{R}^\times$ and L is the complexification of a *real* null line in $\mathbb{R}^{m+n}$ with $\rho L \notin L^\perp$ (thus $\overline{L} = L$),[4] or,
   - $\alpha \in i\mathbb{R}^\times$ and L is the complexification of a null line in $\mathbb{R}^m \oplus i\mathbb{R}^n$ with $\rho L \notin L^\perp$ (thus $\overline{L} = \rho L$).

   The simple factors can then be written

$$p_{\alpha,L}(z) = \frac{\alpha - z}{\alpha + z}P_+ + P_0 + \frac{\alpha+z}{\alpha-z}P_-,$$

[4]Recall that $\mathbb{R}^m, \mathbb{R}^n$ have signature so that this may or may not be possible.

where $P_j : \mathbb{C}^{m+n} \longrightarrow L_j$ is projection onto the vector subspace L_j defined by,

$$L_+ := L, \qquad L_- := \rho L, \qquad L_0 := (L_+ \oplus L_-)^\perp.$$

The dressing action of $p_{\alpha,L}$ is also easy to express in this representation, for $\xi = P_+ - P_-$ and so (4.7) yields

$$p_{\alpha,L} \# \psi = \psi + \frac{2}{\alpha}(\hat{P}_+ - \hat{P}_-), \tag{4.8}$$

where the $\hat{P}$ are projections onto $\hat{L}_+ := \Phi_\alpha^{-1} L_+, \hat{L}_- := \tau\Phi_\alpha^{-1} L_+ = \Phi_{-\alpha}^{-1} L_-$, etc. These are the simple factors and dressing action identified by Burstall in [11].

2. If exceptionally $\mathbb{R}^m \cong \mathbb{R}^n$ (i.e. $m = n$ with the same or opposite signature metrics), then there exist further simple factors. Choose a scalar $\beta \in \mathbb{R}^\times \cup i\mathbb{R}^\times$ and a maximal isotropic subspace V where,
   - $\beta \in \mathbb{R}^\times$ with V the complexification of a *real* maximal isotropic subspace of $\mathbb{R}^{m+n}$ and $\rho V \cap V = \{0\}$ (this requires *opposite* signatures on $\mathbb{R}^m, \mathbb{R}^n$ i.e. $\mathbb{R}^{2,1}, \mathbb{R}^{1,2}$) , or,
   - $\beta \in i\mathbb{R}^\times$ with V the complexification of a maximal isotropic subspace of $\mathbb{R}^m \oplus i\mathbb{R}^n$ and $\rho V \cap V = \{0\}$ (this requires the *same* signature on $\mathbb{R}^m, \mathbb{R}^n$).

   We therefore get simple factors

   $$q_{\beta,V}(z) = \sqrt{\frac{\beta - z}{\beta + z}} Q_+ + \sqrt{\frac{\beta + z}{\beta - z}} Q_-,$$

   where $Q_\pm : \mathbb{C}^{m+n} \longrightarrow V_\pm$ is projection onto $V_+ := V,\ V_- := \rho V_+$ respectively. Again the action of these simple factors can easily be written down, for $\xi = \frac{1}{2}(Q_+ - Q_-)$ so that,

   $$q_{\beta,V} \# \psi = \psi + \frac{1}{\beta}(\hat{Q}_+ - \hat{Q}_-),$$

   with $\hat{Q}_+ : \mathbb{C}^{m+n} \longrightarrow \Phi_\beta^{-1} V_+,\ \hat{Q}_- : \mathbb{C}^{m+n} \longrightarrow \rho\Phi_\beta^{-1} V_+$ projections.

Since Burstall was only concerned with $\mathbb{R}^{m+n} = \mathbb{R}^p \oplus \mathbb{R}^{1,1},\ p \geq 2$, the second list of simple factors was of no relevance. The distinct construction of the two families of simple factors reflects the existence of disjoint conjugacy classes of height 1 parabolic subalgebras of $\mathfrak{so}(2n, \mathbb{C})$: stabilisers of null lines in $\mathbb{C}^{2n}$ and the stabilisers of maximal isotropic planes. The stabilisers of maximal isotropic planes form a single $\mathrm{O}(2n, \mathbb{C})$-conjugacy class but two distinct $\mathrm{SO}(2n, \mathbb{C})$-classes. These classes correspond to the three extreme roots on the Dynkin diagram of $\mathfrak{so}(2n,\mathbb{C})$, all of which have weight 1 in the expansion of the highest root. This is in contrast to the Dynkin diagram of $\mathfrak{so}(2n+1)$ which has only a single root of weight 1 and ties in with the fact the we only see the first list of simple factors when d is odd. These examples will be discussed more fully in chapters 5 and 6.

A similar story to can be told regarding the symplectic group. Under the isomorphism (3.8) we identify the symplectic Lie algebra $\mathfrak{sp}(n,\mathbb{C})$ with the symmetric square of $\mathbb{C}^{2n}$. In contrast to $\mathfrak{so}(2n)$, the Lie algebra $\mathfrak{sp}(n,\mathbb{C})$ has just a single simple root of weight 1 and so we should only expect a single family of simple factors. If, upon choosing a basis, we let ρ be the matrix of the symplectic form ω, then the only possible simple factors are those of the form q above: let $\beta^2 \in \mathbb{R}^\times$, let V be an n-dimensional subspace of $\mathbb{C}^{2n}$ with $\rho V \cap V = \{0\}$ and define

$$q_{\beta,V}(z) = \sqrt{\frac{\beta - z}{\beta + z}}Q_+ + \sqrt{\frac{\beta + z}{\beta - z}}Q_-,$$

with $Q_+ : \mathbb{C}^{m+n} \to V$, $Q_- : \mathbb{C}^{m+n} \to \rho V$ projections. No distinguished choice of real form of $\mathfrak{sp}(n,\mathbb{C})$ has been made: it is perfectly valid to impose a reality condition with respect to a pseudo-symplectic algebra $\mathfrak{sp}(a,b)$. The restrictions implied by such a choice to the subspace V are exactly as those described above for the orthogonal algebra. While it is not surprising that we have only one family of simple factors for $\mathrm{Sp}(n)$, that it should correspond to the second orthogonal family is perhaps mysterious. The reason lies in the difference between the isomorphisms (3.7,3.8): for the orthogonal group the vanishing of $\bigwedge^2 L$ is crucial when L is a line, while the symmetric square of a line, of course, has dimension one.

4.6 Bäcklund Transforms of O-surfaces

In this section we discuss the *Bäcklund transform for O-surfaces* of Schief–Konopelchenko [53] and its relation to the dressing of $\mathfrak{p}$-flat maps by simple factors. We show that their transform is equivalent to the dressing of suitably well-behaved $\mathfrak{p}$-flat maps into $\mathfrak{so}(a,b)$ by simple factors of the first type discussed in Section 4.5.

Definition 4.6
Let $\mathbb{R}^m, \mathbb{R}^n$ have non-degenerate metrics. A map of dual O-surfaces *is a map $\mathbf{R} : \Sigma^l \to \mathbb{R}^m \otimes \mathbb{R}^n$ ($l \leq \min(m,n)$) with co-ordinates $x_1, \ldots, x_l$ such that each $\mathbf{R}_{x_i}$ is of rank one and $\mathbf{R}_{x_i}\mathbf{R}_{x_j}^T = 0 = \mathbf{R}_{x_i}^T\mathbf{R}_{x_j}$ for $i \neq j$.*

It follows that there exist mutually orthogonal frames[5] $\underline{X}_i : \Sigma \to \mathbb{R}^m$, $\overline{H}_i : \Sigma \to \mathbb{R}^n$ such that $\mathbf{R}_{x_i} = \underline{X}_i\overline{H}_i$ for each $i = 1, \dots, l$. It is no condition to assume that $|\underline{X}_i|^2 \in \{0, \pm 1\}$ and by scaling the co-ordinate functions suitably we may also restrict ourselves to $|\overline{H}_i|^2 \in \{0, \pm 1\}$. With the assumption that none of the $\underline{X}_i$ have zero norm it is easy to see that there exist functions $p_{ij} : \Sigma \to \mathbb{R}$ such that

$$(\underline{X}_i)_{x_j} = p_{ij}\underline{X}_j, \quad (\overline{H}_j)_{x_i} = p_{ij}\overline{H}_i, \quad i \neq j.$$

If any of the $\underline{X}_i$ are null then we demand the existence of p_{ij} as above. If we choose fixed bases $\{e_i\}, \{\hat{e}_j\}$ on $\mathbb{R}^m, \mathbb{R}^n$ and contract $\mathbf{R}$ on the left/right respectively we get explicit equations for n surfaces (some of which may be constants) $f^i := \mathbf{R}\hat{e}_i$ in $\mathbb{R}^m$ and m surfaces $g^i = \mathbf{R}^T e_i$ in $\mathbb{R}^n$. Observe that the f^i are Combescure transforms of each other (the tangent vectors for each surface along each co-ordinate x_i are parallel). Similarly the g^i are Combescure related. When all derivatives are non-zero we clearly have conjugate equations on each surface[6]

$$\begin{aligned} f^i_{x_j x_k} &= p_{kj}\frac{(\overline{H}_j, \hat{e}_i)}{(\overline{H}_k, \hat{e}_i)} f^i_{x_j} + p_{jk}\frac{(\overline{H}_k, \hat{e}_i)}{(\overline{H}_j, \hat{e}_i)} f^i_{x_k} \\ &= ((\ln \overline{H}_j)_{x_k}, \hat{e}_i) f^i_{x_j} + ((\ln \overline{H}_k)_{x_j}, \hat{e}_i) f^i_{x_k}, \\ g^i_{x_j x_k} &= ((\ln \underline{X}_j)_{x_k}, e_i) g^i_{x_j} + ((\ln \underline{X}_k)_{x_j}, e_i) g^i_{x_k}. \end{aligned}$$

The co-ordinates x_j are therefore curvature line co-ordinates on each f^i, g^i.

Theorem 4.7 (Schief–Konopelchenko)
Given a solution $(\underline{M}, \underline{N}, S, \beta) : \Sigma \to \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ to the linear system

$$\begin{pmatrix} \underline{M} \\ \underline{N} \end{pmatrix}_{x_i} = \begin{pmatrix} 0 & \beta \underline{X}_i \overline{H}_i \\ \overline{H}_i^T \underline{X}_i^T & 0 \end{pmatrix} \begin{pmatrix} \underline{M} \\ \underline{N} \end{pmatrix} \tag{4.9}$$

satisfying the compatible condition

$$S = \frac{1}{2}|\underline{M}|^2 = \frac{\beta}{2}|\underline{N}|^2, \tag{4.10}$$

we set

$$\hat{\mathbf{R}} = \mathbf{R} - \frac{\underline{M}\underline{N}^T}{S}. \tag{4.11}$$

[5]Viewed as column vectors ($\underline{X}_i$) and row vectors ($\overline{H}_i$) respectively. Note that transpose is always with respect to the metrics on $\mathbb{R}^m, \mathbb{R}^n$.
[6]$\ln \underline{X}_i$ is the vector found by taking the logarithm of the entries of $\underline{X}_i$ with respect to the basis $\{e_i\}$.

Then $\hat{\mathbf{R}}$ is another map of dual O-surfaces into $\Sigma \to \mathbb{R}^m \otimes \mathbb{R}^n$ with respect to the same co-ordinates x_i.

Schief–Konopelchenko prove the above theorem for $m = 3, l = 2$ but the proof is not restricted by dimension. Their proof amounts to an application of the *Fundamental transform* of Eisenhart [28]. We will prove the result indirectly by showing that the transformation corresponds to dressing $\mathfrak{p}$-flat maps by simple factors. That the above should be called a Bäcklund transform is not immediately obvious, the name is however justified by Schief–Konopelchenko when they demonstrate that the classical Bäcklund transform of pseudospherical surfaces in $\mathbb{R}^3$ can be described in terms of Theorem 4.7.

Let $G/K = O(m+n)/O(m) \times O(n)$ with symmetric involution $\tau = \mathrm{Ad}\,\rho$ where ρ has ± 1-eigenspaces $\mathbb{R}^m, \mathbb{R}^n$ respectively. We therefore have $\mathfrak{p} = \mathbb{R}^m \wedge \mathbb{R}^n$. If $\mathbf{R}$ is a map of dual O-surfaces then we can easily build a $\mathfrak{p}$-flat map ψ by

$$\psi = \begin{pmatrix} 0 & \mathbf{R} \\ -\mathbf{R}^T & 0 \end{pmatrix}.$$

That ψ is $\mathfrak{p}$-flat is clear from the orthogonality of the vectors $\underline{X}_i$ and $\overline{H}_i$. The converse however is not necessarily true, for a general $\mathfrak{p}$-flat map may not posses co-ordinates x_j satisfying the definition of a dual O-surface. Observe that when ψ comes from a map of dual O-surfaces such that none of the $\underline{X}_i, \overline{H}_i$ are null then each tangent space $\mathrm{d}\psi(T_s\Sigma)$ is an Abelian, semisimple (diagonalisable) subalgebra of $\mathfrak{p}$. Conversely it is easy to see that $\mathrm{ad}(p \wedge q)$ is 4-step nilpotent if q is null. We show that every $\mathfrak{p}$-flat map with diagonalisable tangent bundle corresponds to a map of dual O-surfaces. Suppose Σ is simply connected. Since ψ is a smooth immersion we can conjugate each tangent space $\mathrm{d}\psi(T_s\Sigma)$ to a fixed semisimple Abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$ (once $\mathfrak{a}$ is fixed this can be done by a unique map $k : \Sigma \to K$). Since $\mathfrak{a}$ is constant we can find co-ordinates x_i for each basis $A_1, \ldots, A_l$ of $\mathfrak{a}$ such that

$$\mathrm{Ad}(k)\,\mathrm{d}\psi = \sum_{i=1}^{l} A_i\,\mathrm{d}x_i. \tag{4.12}$$

The choices of maximal semisimple Abelian algebras of $\mathfrak{p} = \mathbb{R}^m \wedge \mathbb{R}^n$ up to conjugacy depends on the signatures of $\mathbb{R}^m, \mathbb{R}^n$ but such $\mathfrak{a}$ are always of dimension $\min(m,n)$: indeed supposing $m \leq n$ it is not difficult to show that

$$\mathfrak{a} = \langle e_i \wedge f_i \rangle_{i=1}^{m}$$

where $\{e_i\}$ is an orthonormal basis[7] of $\mathbb{R}^m$ and $\{f_i\}$ are a linearly independent orthonormal set in $\mathbb{R}^n$—just use the fact that any $T \in \mathfrak{a}$ has T^2 self-adjoint on $\mathbb{R}^{m+n}$ in order to see the orthogonality. Up to conjugacy the only freedom we have is over how many and which of the f_i have positive norm. The options depend entirely on the signature of $\mathbb{R}^m$.

If however ψ is a $\mathfrak{p}$-flat map with nilpotents in its tangent bundle, there is no guarantee that suitable co-ordinates exist. It is clear that any $\mathfrak{p}$-flat map of a higher dimension than $\min(m,n)$ automatically fails, but for lower dimensions we make no claims. An example of a dual O-surface with non-diagonalisable tangent spaces is the following:

$$\mathbf{R}(x,y) = \begin{pmatrix} x & 0 & 0 \\ 0 & y & y \end{pmatrix} \in \mathbb{R}^2 \otimes \mathbb{R}^{2,1}.$$

Even though this represents a map of dual O-surfaces, the tangent subalgebra contains nilpotents since $\overline{H}_2 = (0,1,1)$ is isotropic.

Now consider the Bäcklund transform of Theorem 4.7. Let ψ be a $\mathfrak{p}$-flat map corresponding to a map of dual O-surfaces $\mathbf{R}$ and let Φ be the extended flat frame of ψ. Given a simple factor $p_{\alpha,L}$ from the first family in Section 4.5, choose a fixed $v \in L$ and define $\underline{M} \in \mathbb{R}^m, \underline{N} \in \mathbb{R}^n$ by

$$v = \Phi_\alpha \begin{pmatrix} \underline{M} \\ \alpha \underline{N} \end{pmatrix}. \tag{4.13}$$

Notice that the conditions on L and α indeed force $\underline{M}, \underline{N}$ to be real vectors. Since v is constant it is easy to see that $\hat{M} := \Phi_\alpha^{-1} v$ satisfies the differential equation

$$d\hat{M} = -\alpha \, d\psi \hat{M}$$

which implies (4.9) with $\beta = -\alpha^2$. Conversely, given $\underline{M}, \underline{N}$ satisfying (4.9), define v by (4.13) with $\alpha = \sqrt{-\beta}$ and observe that v is constant. (4.10) is then precisely the condition that v is isotropic. Solutions $\underline{M}, \underline{N}, -\alpha^2$ are therefore equivalent to simple factors $p_{\alpha,L}$. Recall (4.8) the dressing action of $p_{\alpha,L}$ on a $\mathfrak{p}$-flat map ψ is

$$p_{\alpha,L} \# \psi = \psi + \frac{2}{\alpha}(\hat{P}_+ - \hat{P}_-)$$

[7] $(e_i, e_j) = \pm\delta_{ij}$.

where $\hat{P}_\pm$ are the orthogonal projections onto $\Phi_\alpha^{-1}L, \tau\Phi_\alpha^{-1}L$ respectively. But $\Phi_\alpha^{-1}v = \hat{M}$ and so we can calculate:

$$\hat{P}_+ = \frac{(\tau\hat{M}, -)}{4S}\hat{M}, \quad \hat{P}_- = \frac{(\hat{M}, -)}{4S}\tau\hat{M}.$$

Therefore

$$\begin{aligned}\hat{P}_+ - \hat{P}_- &= \frac{1}{4S}\left(\begin{pmatrix} M \\ \alpha\underline{N}\end{pmatrix}\begin{pmatrix}\underline{M}^T & -\alpha\underline{N}^T\end{pmatrix} - \begin{pmatrix} M \\ -\alpha\underline{N}\end{pmatrix}\begin{pmatrix}\underline{M}^T & \alpha\underline{N}^T\end{pmatrix}\right)\\ &= -\frac{\alpha}{2S}\begin{pmatrix} 0 & \underline{MN}^T \\ -\underline{NM}^T & 0\end{pmatrix},\end{aligned}$$

and so

$$p_{\alpha,L}\#\psi = \psi - \frac{1}{S}\begin{pmatrix} 0 & \underline{MN}^T \\ -\underline{NM}^T & 0\end{pmatrix}$$

which gives rise to a new map $\hat{\mathbf{R}}$ exactly as in (4.11). Consider the dressed simple factor $\hat{p} : \Sigma \to \mathcal{G}^-$. Since $\hat{p}$ is real and twisted we have $\hat{p}(\infty) \in \mathrm{O}(m) \times \mathrm{O}(n)$. (4.6) then tells us that there exist $P \in \mathrm{O}(m)$, $Q \in \mathrm{O}(n)$ such that

$$\mathrm{d}\hat{\mathbf{R}} = P\mathrm{d}\mathbf{R}Q^{-1},$$

from which we see that $\hat{\mathbf{R}}$ is a map of dual O-surfaces. Indeed

$$\hat{\mathbf{R}}_{x_i} = P\underline{X}_i\overline{H}_iQ^{-1} = (P\underline{X}_i)(Q\overline{H}_i^T)^T$$

giving us the new orthogonal families of tangent vectors $P\underline{X}_i$, $\overline{H}_iQ^{-1}$ for free.

Notice finally that since Schief–Konopelchenko parametrise the 1-parameter family of solutions to Theorem 4.7 by $\beta = -\alpha^2$, our previously observed duality $p_{\alpha,L} = p_{-\alpha,\tau L}$ is avoided.

One may obtain a similar identification for the dressing action of the simple factors $q_{\alpha,V}$ of the second type in Section 4.5 when $\mathbb{R}^m, \mathbb{R}^n$ have compatible metrics. The working is however somewhat messy since one must choose a basis of V and define *matrices* $\underline{M}, \underline{N}$ in the manner of (4.13). While the working may be less aesthetically pleasing one can easily obtain an equivalent to Theorem 4.7.

The correspondence of the Bäcklund transform for dual O-surfaces and the dressing of a subset of $\mathfrak{p}$-flat maps by simple factors can allow us to easily calculate the new transformed O-surfaces without having to solve equations (4.9,4.10). We need only to calculate the extended flat frame Φ of the corresponding $\mathfrak{p}$-flat map and then use (4.13) to obtain $\underline{M}, \underline{N}$ satisfying Theorem 4.7. Happily the extended flat frame can be found by exponentiation for very simple $\mathfrak{p}$-flat maps (if $[\psi, d\psi] = 0$ then $\Phi(z) = \exp(z\psi)$) thus making the entire process algebraic. As already noted, once an extended flat frame is known, we may algebraically calculate any transform of ψ by simple factors and therefore any Bäcklund transforms of the underlying O-surfaces.

To illustrate this we provide a couple of examples. Starting with the seed O-surfaces[8] $f^1 = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$, $f^2 = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}$ in $\mathbb{R}^3$ we let $\mathbf{R} = \begin{pmatrix} x & 0 \\ 0 & y \\ 0 & 0 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^3 \otimes \mathbb{R}^{1,1}$ where the metric on $\mathbb{R}^{1,1}$ has matrix $\operatorname{diag}(1, -1)$ with respect to the implied basis. $\mathbf{R}$ thus generates the $\mathfrak{p}$-flat map ψ with extended flat frame Φ as follows:

$$\psi = \begin{pmatrix} 0 & 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 & 0 \\ -x & 0 & 0 & 0 & 0 \\ 0 & y & 0 & 0 & 0 \end{pmatrix}, \quad \Phi_\lambda = \begin{pmatrix} \cos\lambda x & 0 & 0 & \sin\lambda x & 0 \\ 0 & \cosh\lambda y & 0 & 0 & \sinh\lambda y \\ 0 & 0 & 1 & 0 & 0 \\ -\sin\lambda x & 0 & 0 & \cos\lambda x & 0 \\ 0 & \sinh\lambda y & 0 & 0 & \cosh\lambda y \end{pmatrix}.$$

Using (4.13) it is not difficult to calculate, for example, the effect of the simple factor $p_{i,L}$ where $L = \langle 1, -1, 1, 2i, i\rangle$ on $\mathbf{R}$ and therefore the O-surfaces $f^{1,2}$. Indeed we see that

$$\hat{\mathbf{R}} = (\hat{f}^1, \hat{f}^2) = \begin{pmatrix} x - \frac{2}{A}(9e^{2x} - e^{-2x}) & \frac{4}{A}(3e^x - e^{-x})(\cos y + \sin y) \\ \frac{4}{A}(3e^x + e^{-x})(\cos y - \sin y) & y + \frac{8}{A}(1 - 2\cos^2 y) \\ -\frac{4}{A}(3e^x + e^{-x}) & \frac{8}{A}(\cos y + \sin y) \end{pmatrix}$$

where $A = 2 + 9e^{2x} + e^{-2x} - 4\sin(2y)$. It is clear that $\hat{f}^1, \hat{f}^2$ involve only minor perturbations from f^1, f^2, but these are enough to make them interesting as figure 4-1 shows. $\hat{f}^1$ is a single 'bubble' grown on the $\mathbf{i}$-axis, while $\hat{f}^2$ is an infinite string of bubbles grown along the $\mathbf{j}$-axis. Both surfaces approach their respective axes as $x \to \pm\infty$.

Another example is the following pair of transforms of the infinite unit cylinder viewed by letting $\mathbf{R} = \begin{pmatrix} \cos x & \cos x \\ \sin x & \sin x \\ y & y \end{pmatrix} \in \mathbb{R}^3 \otimes \mathbb{R}^{1,1}$. Since $(\overline{H}_1 = \overline{H}_2 = (1,1))$ we have a rather degenerate $\mathfrak{p}$-flat map (tangent spaces are certainly not conjugate to a semi-simple Abelian subalgebra of $\mathfrak{p}$). We dress by a simple factor similar to the previous example (even large changes in $p_{\alpha,L}$ do not change the inherent properties of the transform in this example). Each cylinder is transformed to a 'squashed torus': the infinite ends of the cylinder being folded back into a single central point as $y \to \pm\infty$. Indeed the co-ordinates x and $\hat{y} := 2\tan^{-1} y$ are 2π-periodic and so the image of the

[8] The fact that the seed surfaces are degenerate is of no concern.

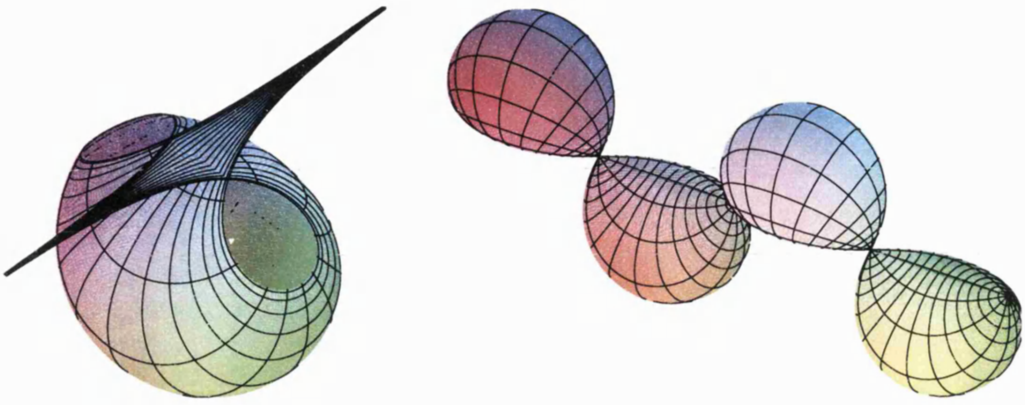

Figure 4-1: The transformed 'bubbles' $\hat{f}_1$ and $\hat{f}_2$

cylinder is a degenerate torus, the central circle having radius 0. The two transforms are actually a reflection of each other across this central limiting point as figure 4-2 suggests.

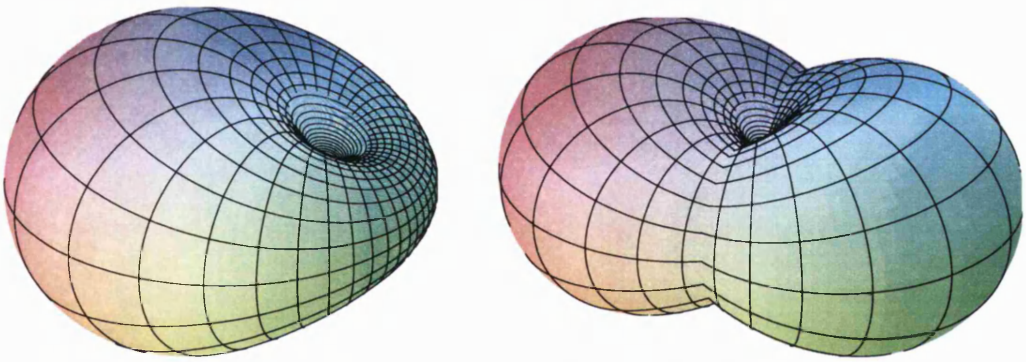

Figure 4-2: The 'squashed tori': $\hat{f}_1$, then $\hat{f}_1$ & $\hat{f}_2$ together

Chapter 5

Isothermic Submanifolds of Symmetric R-spaces

5.1 Introduction

The theory of isothermic submanifolds lies at the intersection of many of the ideas already discussed in this thesis. We extend the theory of isothermic surfaces in the conformal n-sphere, as developed by Burstall [11], by observing that the definition of an isothermic surface $\ell : \Sigma^2 \to S^n = \mathbb{P}(\mathcal{L}^{n+1,1})$ requires nothing more than the existence of a closed 1-form η taking values in the nilradical of the parabolic stabiliser of ℓ (as a subalgebra of $\mathfrak{so}(n+1,1)$). The crucial algebraic structure possessed by S^n is that of a symmetric R-space: a conjugacy class of parabolic subalgebras of height 1. We give a definition of isothermic submanifold in any such space and demonstrate that the transforms of the original theory (Christoffel, T- and Darboux) are all available, along with the interactions between them. The new theory is somewhat more pleasing than the old due to a new bundle-approach that allows us to work directly with the submanifolds themselves rather than with choices of 'frames' as in [11]. Symmetric R-spaces come in two flavours, self-dual (e.g. the motivational $\mathbb{P}(\mathcal{L}^{n+1,1})$ as an $\mathrm{SO}(n+1,1)$-space) and non-self-dual (e.g. the Grassmannians $G_k(\mathbb{R}^n)$, $k \neq n/2$ as $\mathrm{SL}(n)$-spaces). The theories for both flavours are almost identical, the only significant difference coming when we discuss the Bianchi permutability of Darboux transforms: in the self-dual case multiple proofs are given, all of which fail in non-self-dual symmetric R-spaces. Symmetric R-spaces also play a secondary role in this chapter, providing a method of generating transforms: the simple factor dressing theory of chapters 3 and 4 translates naturally to a dressing theory of curved flats. When applied to curved flats representing Darboux pairs of isothermic submanifolds, we obtain a second proof of the Bianchi permutability theorem.

5.2 Isothermic surfaces in the Conformal n-sphere

Recall isothermic surfaces in $\mathbb{R}^n$. An immersion $f : \Sigma^2 \to \mathbb{R}^n$ is isothermic iff there exist conformal curvature line co-ordinates x, y: i.e. the first fundamental form is conformal to $\mathrm{d}x^2 + \mathrm{d}y^2$, while $\mathbb{I}$ is diagonal. It follows that the 1-form

$$\omega = -\frac{1}{|f_x|^2} f_x \,\mathrm{d}x + \frac{1}{|f_y|^2} f_y \,\mathrm{d}y$$

is closed and so we can locally integrate $\omega = \mathrm{d}\tilde{f}$ to find a *Christoffel transform* $\tilde{f} : \Sigma^2 \to \mathbb{R}^n$. $\tilde{f}$ has parallel curvature lines to f but with opposite orientation: $\mathrm{d}f^{-1} \circ \mathrm{d}\tilde{f} \in \operatorname{End} T\Sigma$ has $\det < 0$. Calculating $\tilde{\omega}$ in the obvious way shows that $\tilde{f}$ is also isothermic with Christoffel transform f. Christoffel transforms are not unique, since any uniform scaling or translation of $\tilde{f}$ differentiates to a multiple of ω.

The geometry of isothermic surfaces in $\mathbb{R}^n$ can be embedded in the conformal n-sphere $S^n \cong \mathbb{P}(\mathcal{L}^{n+1,1})$ by an inverse stereo-projection into the light cone $\mathcal{L}^{n+1,1} \subset \mathbb{R}^{n+1,1}$. Fix $\mathbb{R}^n \subset \mathbb{R}^{n+1,1}$ and choose $t \in (\mathbb{R}^n)^\perp$ such that $|t|^2 = -1$. Pick a unit $n \in (\langle t \rangle \oplus \mathbb{R}^n)^\perp$ (there are only two). The inverse stereo-projection of f with respect to n is

$$\frac{2}{1+|f|^2} f - \frac{1-|f|^2}{1+|f|^2} n \in S^n \subset \langle t \rangle^\perp,$$

which can be lifted into the light cone by adding t. In this way we have a diffeomorphism $S^n \cong \mathbb{P}(\mathcal{L}^{n+1,1})$ and can furthermore lift isothermic surfaces in $\mathbb{R}^n$ to $\mathbb{P}(\mathcal{L}^{n+1,1})$. To make contact with the discussion in [11] define $v_0 = \frac{1}{2}(t-n)$, $v_\infty = \frac{1}{2}(t+n)$ so that the lift of f into the projective light cone is

$$\Lambda := \left\langle f + v_0 + |f|^2 v_\infty \right\rangle.$$

The points v_0, v_∞ are the 'points at zero and ∞' respectively for the stereo-projection map. We call Λ an isothermic surface iff f is isothermic in $\mathbb{R}^n$. Under the usual identification (e.g. (3.7))

$$\bigwedge^2 \mathbb{R}^{n+1,1} \cong \mathfrak{so}(n+1,1) \tag{5.1}$$

it is easy to see that

$$\Lambda = \langle \exp(2v_\infty \wedge f) v_0 \rangle.$$

Let us see what inverse stereo-projection does to the closed 1-form $\omega = \mathrm{d}\tilde{f}$. Define[1]

$$\eta := \exp(2v_\infty \wedge f)(v_0 \wedge \omega) \in \Omega^1_\Sigma \otimes \Lambda \wedge \Lambda^\perp. \tag{5.2}$$

Then, writing $\dot{\wedge}$ for wedge product of differential forms to distinguish from that of vectors, we have

$$\mathrm{d}\eta = \exp(2v_\infty \wedge f)\Big(v_0 \wedge \mathrm{d}\omega + 2[(v_\infty \wedge \mathrm{d}f) \,\dot{\wedge}\, (v_0 \wedge \omega)]\Big) \tag{5.3}$$

Evaluating on vector fields $X, Y \in \Gamma T\Sigma$ we see that

$$\begin{aligned}[(v_\infty \wedge \mathrm{d}f) \,\dot{\wedge}\, (v_0 \wedge \mathrm{d}\tilde{f})](X,Y) &= ((\mathrm{d}f_X, \mathrm{d}\tilde{f}_Y) - (\mathrm{d}f_Y, \mathrm{d}\tilde{f}_X))v_\infty \wedge v_0 \\ &\quad + (v_0, v_\infty)(\mathrm{d}f_X \wedge \mathrm{d}\tilde{f}_Y - \mathrm{d}f_Y \wedge \mathrm{d}\tilde{f}_X) \\ &= 0 \iff \mathrm{d}f \,\dot{\wedge}_{C\ell}\, \mathrm{d}\tilde{f} = 0,\end{aligned}$$

where $\dot{\wedge}_{C\ell}$ is evaluated in the Clifford algebra $C\ell(\mathbb{R}^n)$. Burstall [11] demonstrated that $f, \tilde{f}$ are a Christoffel pair in $\mathbb{R}^n$ iff $\mathrm{d}f \,\dot{\wedge}_{C\ell}\, \mathrm{d}\tilde{f} = 0$ and so η is closed.

A Christoffel pair therefore defines a closed 1-form η with values in $\Lambda \wedge \Lambda^\perp$. The converse is also true: given any 1-form η with values in $\Lambda \wedge \Lambda^\perp$, there exists an $\omega \in \Omega^1_\Sigma \otimes \mathbb{R}^n$ satisfying (5.2). (5.3) says that η is closed iff ω is closed and $\mathrm{d}f \,\dot{\wedge}_{C\ell}\, \omega = 0$. We can therefore make the following definition.

Definition 5.1
An immersion $\Lambda : \Sigma^2 \to \mathbb{P}(\mathcal{L}^{n+1,1})$ is isothermic iff $\exists$ closed $\eta \in \Omega^1_\Sigma \otimes \Lambda \wedge \Lambda^\perp$.

By reversing the choice of points at 0 and ∞ we can define Christoffel transforms in this setting. Let $f, \tilde{f}$ be a Christoffel pair, where f is the stereo-projection (with respect to some v_0, v_∞) of the isothermic surface (Λ, η). Then

$$\tilde{\Lambda} := \langle \exp(2v_0 \wedge \tilde{f})v_\infty \rangle \tag{5.4}$$

is a Christoffel transform of Λ. $\tilde{\Lambda}$ is isothermic with closed 1-form $\exp(2v_0 \wedge \tilde{f})(v_\infty \wedge \mathrm{d}f)$. In this context we could have defined $\tilde{\Lambda} = \langle \exp(2v_0 \wedge \tilde{f})v_0 \rangle$ which is also isothermic. It will later be seen that Christoffel transforms in a general setting must inhabit a 'dual' space to that of Λ and that (5.4) is the correct definition. Christoffel transforms in the projective light cone are far more freely available than in $\mathbb{R}^n$ since we now have a free choice of v_0, v_∞ with respect to which we stereo-project.

[1] $\Lambda^\perp$ is the subbundle of $\Sigma \times \mathbb{R}^{n+1,1}$ whose fibres consist of the vectors perpendicular to the line Λ. $\Lambda \wedge \Lambda^\perp$ is then a subbundle of $\Sigma \times \bigwedge^2 \mathbb{R}^{n+1,1}$.

5.3 Symmetric R-spaces

After the identifications of the previous section, Definition 5.1 is remarkably simple to state. An isothermic surface depends only on the existence of a closed 1-form with values in a location that depends on the surface.We can abstract the definition further by generalising the target space $\mathbb{P}(\mathcal{L}^{n+1,1})$. Consider $\mathbb{P}(\mathcal{L}^{n+1,1})$ as a homogeneous $\mathrm{SO}(n+1,1)$-space: under the identification (5.1) the Lie algebra of the stabiliser of Λ is seen to be

$$\mathfrak{p} = \Lambda \wedge \Lambda^{\perp} \oplus \bigwedge^2 \Lambda^{\perp} \oplus \Lambda \wedge \hat{\Lambda},$$

where $\hat{\Lambda}$ is any null line such that $\Lambda \neq \hat{\Lambda}$. The Killing form on $\mathfrak{so}(n+1,1)$ is conformal to the trace $\mathrm{tr}(\,\cdot\,\cdot\,)$, which makes it easy to see that $\mathfrak{p}^{\perp} = \Lambda \wedge \Lambda^{\perp}$. Observe that $\mathfrak{p}$ is a bundle of parabolic subalgebras (Definition 1.9) of $\mathfrak{so}(n+1,1)$ with Abelian nilradical $\mathfrak{p}^{\perp} = \Lambda \wedge \Lambda^{\perp}$.

Definition 5.2
A subgroup P of a (real or complex) semisimple Lie group G is parabolic iff it is the stabiliser of a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$. Since $\mathfrak{p}$ is self-normalising (Lemma 1.12) it is clear that $\mathfrak{p}$ is the Lie algebra of P. An R-space M is a conjugacy class of parabolic subalgebras: i.e. $M = G/P$ where P is parabolic. A symmetric R-space is an R-space for which $\mathfrak{p}$ has Abelian nilradical.

The reader is referred to the discussion of Section 1.3 on which the results of this chapter rely heavily.

A symmetric R-space M is, as its name suggests, also a symmetric space (Definition 1.5). For real M it can be shown (see e.g. [42, Theorem A]) that the maximal compact subgroup $\tilde{G} \subset G$ acts transitively on G/P with stabiliser $K := \tilde{G} \cap P$ and that $\tilde{G}/K$ is a symmetric space: i.e. the Lie algebras $\tilde{\mathfrak{g}}, \mathfrak{k}$ of $\tilde{G}, K$ describe a symmetric decomposition $\tilde{\mathfrak{g}} = \mathfrak{k} \oplus \mathfrak{k}^{\perp}$. Conversely [42, Theorems A and B], given any symmetric space $M = \tilde{G}/K$ where $\tilde{G}$ is compact, *if* there exists a finite dimensional Lie group G of diffeomorphisms of $\tilde{G}/K$ which is strictly larger than $\tilde{G}$, then $\tilde{G}/K$, viewed as a homogeneous G-space, is a symmetric R-space. It follows that the only R-spaces that can also be viewed as symmetric are the symmetric R-spaces and so the terminology is consistent. Similarly complex M are seen to be Hermitian symmetric spaces.

Classification of symmetric R-spaces is easy in view of the discussion of Section 1.3. It is straightforward to see that the restriction of a parabolic subalgebra of $\mathfrak{g}$ to a simple ideal is a parabolic subalgebra of that ideal (of lower or equal height). It follows that a general R-space is the direct product of R-spaces G/P where G is simple.

The height of an R-space is then the largest of the heights of the simple terms in the product. The definitive list of *symmetric* R-spaces G/P where G is simple is given in Theorem B of [42].

The following are examples of symmetric R-spaces with their parabolic stabilisers: we have underlined the Abelian nilradicals.

$$\mathbb{P}(\mathcal{L}^{p+1,q+1}) = \frac{S^p \times S^q}{\mathbb{Z}_2} = \frac{\mathrm{SO}(p+1,q+1)}{P} = \frac{\mathrm{SO}(p+1)\times \mathrm{SO}(q+1)}{\mathrm{SO}(p)\times\mathrm{SO}(q)\times\mathbb{Z}_2}$$

$$\mathfrak{p} = \underline{\Lambda\wedge\Lambda^\perp} \oplus \textstyle\bigwedge^2\Lambda \oplus \Lambda\wedge\hat{\Lambda} \qquad (\hat{\Lambda}\in\mathcal{L}\cap(\mathbb{R}^n/\Lambda^\perp)),$$

$$G_k(\mathbb{R}^n) = \frac{\mathrm{SL}(n)}{P} = \frac{\mathrm{SO}(n)}{\mathrm{S}(\mathrm{O}(k)\times\mathrm{O}(n-k))}$$

$$\mathfrak{p} = \underline{\mathrm{hom}(\mathbb{R}^n/\pi,\pi)} \oplus (\mathrm{End}(\pi)\oplus\mathrm{End}(\mathbb{R}^n/\pi))_0,$$

$$\mathrm{SO}(n) = \frac{\mathrm{SO}(n,n)}{P} = \frac{\mathrm{SO}(n)\times\mathrm{SO}(n)}{\mathrm{SO}(n)}$$

$$\mathfrak{p} = \underline{\textstyle\bigwedge^2\varphi} \oplus \varphi\wedge\hat{\varphi} \qquad (\mathbb{R}^{n,n} = \varphi\oplus\hat{\varphi} \text{ as isotropic } n\text{-planes}).$$

The second and third examples will be discussed in greater detail in Chapter 6. In particular it will be demonstrated that $\mathrm{SO}(n)$ is indeed a symmetric R-space viewed as exactly half the set of isotropic n-planes in $\mathbb{R}^{n,n}$.

We can now make a general definition of an isothermic submanifold.

Definition 5.3

An isothermic submanifold of a (real or complex) symmetric R-space G/P is an immersion f of a real manifold Σ into G/P for which there exists a closed $\eta \in \Omega^1_\Sigma \otimes f^\perp$.

Remarks: An isothermic submanifold f is a bundle of parabolic subalgebras over Σ and as such will be viewed as a subbundle of the trivial Lie algebra bundle $\underline{\mathfrak{g}} = \Sigma\times\mathfrak{g}$. The notation $\eta\in\Omega^1_\Sigma\otimes f^\perp$ means that the closed 1-form takes values in the *bundle* of nilradicals $f^\perp$. *Any* immersion of a 1-manifold into a symmetric R-space is trivially isothermic.

We place no restrictions on the dimension of Σ: the maximum dimension of an isothermic submanifold depends on the symmetric R-space in question. For the above examples the maximum practical dimensions are 2, $\min(k, n-k)$, $\lfloor n/2\rfloor$ respectively although a proof, and indeed what we mean by 'practical', will have to wait until Section 5.7 when we consider curved flats.

Dual R-spaces

Let M be an R-space. The dual R-space M^* is the set of parabolic subalgebras complementary to any element of M. The next proposition tells us that M^* is indeed an R-space.

Proposition 5.4
M^* is a conjugacy class of parabolic subalgebras. Indeed if $(\mathfrak{p}, \mathfrak{q})$ are complementary for $\mathfrak{p} \in M$, then M^* is the conjugacy class of $\mathfrak{q}$.

Proof $\tilde{\mathfrak{q}} = \mathrm{Ad}\, g\mathfrak{q}$ is complementary to $\mathrm{Ad}\, g\mathfrak{p} \in M$ and thus $\tilde{\mathfrak{q}} \in M^*$. Conversely, if $(\mathrm{Ad}\, g\mathfrak{p}, \hat{\mathfrak{q}})$ are complementary, then $\mathrm{Ad}(g^{-1})\hat{\mathfrak{q}}$ is complementary to $\mathfrak{p}$ and so, by Proposition 1.13, $\exists$ unique $n \in \mathfrak{p}^{\perp}$ such that,

$$\mathrm{Ad}(g^{-1})\hat{\mathfrak{q}} = \mathrm{Ad}\exp n\mathfrak{q} \Rightarrow \hat{\mathfrak{q}} = \mathrm{Ad}(g \exp n)\mathfrak{q},$$

which is in the conjugacy class of $\mathfrak{q}$.

■

Since complementary parabolic subalgebras have the same height it is immediate that M^* is a *symmetric* R-space iff M is. The dual R-space can be viewed as a homogeneous space $M^* = G/Q$ where Q is the parabolic subgroup of G with Lie algebra $\mathfrak{q}$, for *any* $\mathfrak{q}$ complementary to $\mathfrak{p}$.

The distinction between self- and non-self-dual M is critically important when it comes to our later discussions of Bianchi permutability of Darboux transforms. An important result with regard to this is the following lemma.

Lemma 5.5
Suppose $K \in \mathrm{Aut}(\mathfrak{g})$ preserves some $f \in M$. Then M is K-invariant.

Proof Let $g \in G$. $\mathrm{Ad}(G)$ is a normal subgroup of $\mathrm{Aut}(\mathfrak{g})$ and so $K\,\mathrm{Ad}(g)f = K\,\mathrm{Ad}(g)K^{-1}Kf = \mathrm{Ad}(h)f \in M$ for some $h \in G$.

■

The above approach of fixing a particular parabolic subalgebra $\mathfrak{p}$ is useful for some calculations but generally unnecessary. The fact that all parabolic subalgebras in G/P are conjugate to some fixed $\mathfrak{p}$ will be used, but from now on we make no special choice of such.

Referring back to our earlier examples we see that a complementary parabolic subalgebra to the stabiliser of a null line in $\mathbb{R}^{p+1,q+1}$ is the stabiliser of a second, distinct null line. $\mathbb{P}(\mathcal{L}^{p+1,q+1})$ is therefore self-dual. A complement to the stabiliser of a k-plane π in $\mathbb{R}^n$ is the stabiliser of an $(n-k)$-plane $\hat{\pi}$ such that $\pi \cap \hat{\pi} = \{0\}$. Thus

$(G_k(\mathbb{R}^n))^* = G_{n-k}(\mathbb{R}^n)$. The $\mathrm{SO}(n,n)$ example is best described in terms of the set of isotropic n-planes in $\mathbb{R}^{n,n}$ which comprises two $\mathrm{SO}(n,n)$-orbits (cf. α and β planes in $\mathbb{R}^{3,3}$ [62]). $\mathfrak{p} \in \mathrm{SO}(n,n)/P$ is the stabiliser of an isotropic n-plane. It is easy to see that $(\mathrm{stab}(\varphi), \mathrm{stab}(\hat{\varphi}))$ are complementary iff $\mathbb{R}^{n,n} = \varphi \oplus \hat{\varphi}$. It will be shown in Section 6.5 that $\hat{\varphi}$ is in the $\mathrm{SO}(n,n)$-orbit of φ iff n is even and so $\mathrm{SO}(n,n)/P$ is self-dual iff n is even.

5.4 The Symmetric Space of Complementary Pairs

Let $f : \Sigma \to M$ be an immersion of a manifold Σ. Since f is a bundle of parabolic subalgebras of $\mathfrak{g}$, the pointwise filtering (1.15) induces a filtering of the trivial Lie algebra bundle

$$\underline{\mathfrak{g}} \supsetneq f \supsetneq f^\perp \supsetneq \{0\} \rightsquigarrow \underline{\mathfrak{g}} = \underline{\mathfrak{g}}/f \oplus f/f^\perp \oplus f^\perp. \tag{5.5}$$

Recalling (1.12) we see that the soldering form β gives an isomorphism of the tangent bundle to M along f

$$\beta : f^*TM \cong \underline{\mathfrak{g}}/f. \tag{5.6}$$

The soldering form β gives yet more structure on a symmetric R-space, for we know (1.5) that the adjoint action of β on $\underline{\mathfrak{g}}$ amounts to differentiation of sections modulo a quotient. Applying (1.5) to the filtering (5.5) gives a 'filtered differentiation' on $\underline{\mathfrak{g}}$:

$$\operatorname{ad}\beta(v) = \begin{cases} \mathrm{d}v \mod f^\perp, & \text{if } v \in \Gamma f^\perp, \\ \mathrm{d}v \mod f, & \text{if } v \in \Gamma f/f^\perp, \\ 0 (= \mathrm{d}v \mod \underline{\mathfrak{g}}), & \text{if } v \in \Gamma \underline{\mathfrak{g}}/f. \end{cases} \tag{5.7}$$

For this to be well-defined we require $\mathrm{d}_X f^\perp \subset f$, but this is clear since any section of $f^\perp$ is $\mathrm{f} = \mathrm{Ad}(g)p$ for some $g : \Sigma \to G$ and $p \in \Gamma\underline{\mathfrak{p}}^\perp$ where $\mathfrak{p}$ is some fixed parabolic subalgebra: therefore

$$\mathrm{df} = \mathrm{Ad}(g)(\mathrm{d}p + [g^{-1}\mathrm{d}g, p]) \in \Omega^1_\Sigma \otimes f$$

since $\mathfrak{p}$ is fixed and $[\mathfrak{g}, \mathfrak{p}^\perp] \subset \mathfrak{p}$.

Now suppose that we have a second bundle of parabolic subalgebras $\hat{f} : \Sigma \to M^*$ such that $(f, \hat{f})$ are always complementary. Call the space of all such pairs Z. Apart from providing a $\hat{\beta} \in \Omega^1_\Sigma \otimes \underline{\mathfrak{g}}/\hat{f}$, the complement defines a grading (Proposition 1.11) of the bundle $\underline{\mathfrak{g}}$ and unique linear isomorphisms

$$\underline{\mathfrak{g}}/f \cong \hat{f}^{\perp}, \qquad f/f^{\perp} \cong f \cap \hat{f} \cong \hat{f}/\hat{f}^{\perp}, \qquad \underline{\mathfrak{g}}/\hat{f} \cong f^{\perp}, \tag{5.8}$$

etc., with respect to which we may view $\beta, \hat{\beta}$ as taking values in $\hat{f}^{\perp}, f^{\perp}$ respectively. While in practice we will nearly always work in the presence of such isomorphisms, the notation usefully reminds us that $\beta, \hat{\beta}$ are actually quotient-valued. The confusion increases if we consider a second complement $\tilde{f} : \Sigma \longrightarrow M^*$ which comes equipped with a $\tilde{\beta} \in \Omega^1_\Sigma \otimes \underline{\mathfrak{g}}/\tilde{f}$ and isomorphisms $\underline{\mathfrak{g}}/f \cong \tilde{f}^{\perp}$, etc., so that $\beta, \tilde{\beta}$ now take values in $\tilde{f}^{\perp}, f^{\perp}$ respectively. One must take care however, for although $\hat{\beta}, \tilde{\beta}$ appear to take values in the same space ($f^{\perp}$), in practice only one set of isomorphisms (5.8) is operational and so there should be no confusion.

Proposition 5.6
The space of complementary pairs Z is symmetric.

Proof Let $(f, \hat{f}), (F, \hat{F}) \in Z$. Supposing that $(F, \hat{f})$ are complementary then by two applications of Proposition 1.13 we see that there exist unique sections $n \in \Gamma\hat{f}^{\perp}$ and $n' \in \Gamma F^{\perp}$ such that

$$(F, \hat{f}) = \operatorname{Ad}\exp n(f, \hat{f}) \quad \text{and} \quad (F, \hat{F}) = \operatorname{Ad}(\exp n' \exp n)(f, \hat{f}).$$

If $(F, \hat{f}) \notin Z$ then, since $C_{\hat{f}}$ is dense in M, there exists $h : \Sigma \longrightarrow G$ such that $(h \cdot F, \hat{f}) \in Z$ and the above argument holds with the insertion of a final $\operatorname{Ad} h^{-1}$. It follows that G acts transitively on Z and so $Z = G/(P \cap Q)$ as a homogeneous space, where $M = G/P$ and $M^* = G/Q$. Define the bundle decomposition

$$\underline{\mathfrak{g}} = \mathfrak{h} \oplus \mathfrak{m} := f \cap \hat{f} \oplus (f^{\perp} \oplus \hat{f}^{\perp})$$

and let τ be the involution of $\underline{\mathfrak{g}}$ with ± 1-eigenspaces $\mathfrak{h}, \mathfrak{m}$ respectively. It is clear that $\underline{\mathfrak{g}} = \mathfrak{h} \oplus \mathfrak{m}$ is a symmetric decomposition and so Z is symmetric with involution τ in the sense of the bundle version of Definition 1.5.

■

One final piece of structure should be observed. Under the isomorphisms (5.8) the adjoint actions of $\beta, \hat{\beta}$ (5.7) on a section $v \in \Gamma\underline{\mathfrak{g}}$ combine to give

$$\operatorname{ad}(\beta + \hat{\beta})(v) = \begin{cases} \operatorname{Proj}_{\mathfrak{m}} dv, & \text{if } v \in \Gamma\mathfrak{h}, \\ \operatorname{Proj}_{\mathfrak{h}} dv, & \text{if } v \in \Gamma\mathfrak{m}. \end{cases}$$

That is $\beta + \hat{\beta} = \mathcal{N}$, the Maurer–Cartan form of Z.

5.5 Christoffel, Darboux and T-Transforms

A notion of stereo-projection akin to that in Section 5.2 is available in any symmetric R-space M. Fix a pair of complementary parabolic subalgebras $(\mathfrak{v}_0, \mathfrak{v}_\infty) \in Z$. $(\mathfrak{v}_0, \mathfrak{v}_\infty)$ define a stereo-projection of the big-cell $C_{\mathfrak{v}_\infty} \subset M$ of parabolic subalgebras complementary to $\mathfrak{v}_\infty$ (cf. Proposition 1.13). Let $f : \Sigma \to C_{\mathfrak{v}_\infty}$: by Proposition 1.13 $\exists$ unique $F_\infty : \Sigma \to \mathfrak{v}_\infty^\perp$ such that

$$f = \operatorname{Ad} \exp F_\infty \mathfrak{v}_0.$$

F_∞ is the *stereo-projection* of f with respect to $\mathfrak{v}_0, \mathfrak{v}_\infty$. Similarly the stereo-projection of $\hat{f} : \Sigma \to C_{\mathfrak{v}_0} \subset M^*$ with respect to $\mathfrak{v}_0, \mathfrak{v}_\infty$ is a map $F_0 : \Sigma \to \mathfrak{v}_0^\perp$. The stereo-projection allows us to build Christoffel transforms. Let $(f, \eta) : \Sigma \to C_{\mathfrak{v}_\infty}$ be isothermic, then there exist unique $F_\infty : \Sigma \to \mathfrak{v}_\infty^\perp$, $\omega \in \Omega^1_\Sigma \otimes \mathfrak{v}_0^\perp$ such that $(f, \eta) = \operatorname{Ad} \exp F_\infty (\mathfrak{v}_0, \omega)$. The closure of η is equivalent to

$$\mathrm{d}\omega = 0 = [\mathrm{d}F_\infty \wedge \omega].$$

Choose a local solution $F_0 : \Sigma \to \mathfrak{v}_0^\perp$ to $\mathrm{d}F_0 = \omega$ and observe that

$$\tilde{f} := \operatorname{Ad} \exp F_0 \mathfrak{v}_\infty$$

is isothermic in M^* with closed 1-form $\tilde{\eta} = \operatorname{Ad} \exp F_0 \, \mathrm{d}F_\infty$. Compare this construction with our earlier definition of Christoffel transform (5.4) in the conformal n-sphere: $\mathfrak{v}_0, \mathfrak{v}_\infty$ are the stabilisers of two distinct null lines $\langle v_0 \rangle, \langle v_\infty \rangle$ respectively and so $\mathfrak{v}_0 = \langle v_0 \rangle \wedge \langle v_0, v_\infty \rangle^\perp$, $\mathfrak{v}_\infty = \langle v_\infty \rangle \wedge \langle v_0, v_\infty \rangle^\perp$. Since v_0, v_∞ are points we are able to view stereo-projections as maps into the same space ($\mathbb{R}^n := \langle v_0, v_\infty \rangle^\perp$) which gives rise to the confusion as to the definition of $\tilde{\Lambda}$. In general symmetric R-spaces it is clear that this confusion cannot arise, for Christoffel pairs are forced to inhabit dual conjugacy classes.

There is a $(\dim Z = 2 \dim M)$-dimensional choice of stereo-projections, each of which (via translation of F_0 by a constant) gives rise to a $(\dim G/P)$-dimensional choice of Christoffel transforms. Since a choice of stereo-projection is equivalent to a choice of fixed base point $(\mathfrak{v}_0, \mathfrak{v}_\infty) \in Z$, all choices of are equivalent up to the left action of G. Even taking this into account, there is still a significant freedom of choice of Christoffel transform. This is in marked contrast to our next transform (the T-transform) which can be defined without any choices whatsoever and will be seen to be unique up to the left action of G.

A recurring theme of this chapter is the idea of transforming a submanifold by gauging flat differentiation while fixing the submanifold. Consequently we will often write an isothermic submanifold as a triple (f, D, η) where D is a flat connection on $\underline{\mathfrak{g}}$ such that $\mathrm{d}^D\eta = 0$. A more general definition of Christoffel transform can be phrased in this form:

Definition 5.7
A pair (f, D, η), $(\tilde{f}, D, \tilde{\eta})$ of isothermic submanifolds is a Christoffel pair iff there exists a pair $(\mathfrak{v}_0, \mathfrak{v}_\infty) \in \Gamma Z$ of D-parallel bundles of parabolic subalgebras such that the stereo-projections $F, \tilde{F}$ of $f, \tilde{f}$ with respect to $(\mathfrak{v}_0, \mathfrak{v}_\infty)$ satisfy $[DF \wedge D\tilde{F}] = 0$.

It should be noted that the new definition does not allow extra Christoffel transforms of an isothermic surface, we are merely viewing existing transforms in a different way: since D, d are flat, there locally exists a gauge transform $\Phi : \Sigma \to G$ satisfying $D = \Phi \circ \mathrm{d} \circ \Phi^{-1}$, then the D- and d-Christoffel transforms differ only by the left action of Φ.

T-transforms

An isothermic submanifold comes equipped with a family of flat G-connections $\mathrm{d}^t := \mathrm{d} + t\eta$ on $\underline{\mathfrak{g}}$. That each d^t is a connection is clear since $\mathrm{d}^t - \mathrm{d} \in \Omega^1_\Sigma \otimes \mathrm{End}(\underline{\mathfrak{g}})$. For the flatness

$$R^{\mathrm{d}^t} = R^{\mathrm{d}} + t\mathrm{d}\eta + \frac{1}{2}[t\eta \wedge t\eta] = 0$$

since d is flat and η is closed with values in the Abelian subalgebra $f^\perp$. It is the fact that flat G-connections integrate to group-valued functions and that an isothermic submanifold comes equipped with a 1-parameter family of such connections that drives most of the interesting geometry of isothermic submanifolds and what makes them an integrable system. Indeed the closed 1-form $t\eta$ satisfies the Maurer–Cartan equations

$$\mathrm{d}(t\eta) + \frac{1}{2}[t\eta \wedge t\eta] = 0 \tag{5.9}$$

for all $t \in \mathbb{R}$ and so we can locally integrate $\Phi_t^{-1}\mathrm{d}\Phi_t = t\eta$ to find a map $\Phi_t : \Sigma \to G$, unique up to the left action of G, which gauges the two connections: $\mathrm{d}^t = \Phi_t^{-1} \circ \mathrm{d} \circ \Phi_t$. Notice that $f_t := \Phi_t f$ is isothermic with closed 1-form $\eta_t := \Phi_t \eta$.

Definition 5.8
f_t is a T-transform *of f and we write $f_t = \mathcal{T}_t f$.*

Equivalently (f,η) is isothermic with d replaced with d^t, so we can take the point of view that f_t is the *same* submanifold as f viewed with respect to a different idea of flat differentiation and succinctly write

$$\mathcal{T}_t(f,\mathrm{d},\eta) = (f,\mathrm{d}^t,\eta).$$

This approach has the advantage of removing choices in the definition of Φ_t: since $\Phi_t' := K\Phi_t$ satisfies the same equation as Φ for any constant $K \in G$, T-transforms as described by solutions to (5.9) are only defined up to the left action of G. Furthermore since $\mathcal{T}_t(f,\eta) = \mathcal{T}_s(f,\frac{t}{s}\eta)$, it is to be understood that η is fixed from the start, otherwise none of the following formulae make sense.

To make contact with older ideas of the T-transform let $f,\tilde{f}$ be a Christoffel pair with respect to $\mathfrak{v}_0,\mathfrak{v}_\infty$ (giving unique F_0,F_∞) such that

$$f = \operatorname{Ad}\exp F_\infty \mathfrak{v}_0, \qquad \tilde{f} = \operatorname{Ad}\exp F_0 \mathfrak{v}_\infty.$$

Let $\Psi_0 = \exp F_\infty$ and define $\Psi_t = \Phi_t\Psi_0$, which has logarithmic derivative

$$\Psi_t^{-1}\mathrm{d}\Psi_t = \mathrm{d}F_\infty + t\operatorname{Ad}\Psi_0^{-1}\eta = \mathrm{d}F_\infty + t\mathrm{d}F_0. \tag{5.10}$$

The T-transforms of f and $\tilde{f}$ are given by

$$\begin{aligned}\mathcal{T}_t f &= \Psi_t\cdot\mathfrak{v}_0 = \Phi_t\cdot f,\\ \mathcal{T}_t\tilde{f} &= \Psi_t\cdot\mathfrak{v}_\infty = \Psi_t K_t\cdot\mathfrak{v}_\infty = \Psi_t K_t\exp(-F_0)\cdot\tilde{f},\end{aligned}$$

where K_t has eigenvalues $t,1,t^{-1}$ on $\mathfrak{v}_0^\perp, \mathfrak{v}_0\cap\mathfrak{v}_\infty, \mathfrak{v}_\infty^\perp$ respectively. It is easy to see that the logarithmic derivative of $\Psi_t K_t\exp(-F_0)$ is $t\tilde{\eta}$ as required.[2] (5.10) is exactly the expression given in the definition of the T-transform in [11]. Consequently the name T-transform is justified, for [11] extends the original definition of the T-transform as introduced by Bianchi [3] and Calapso [19, 20]. In comparison to Burstall's construction, our approach is relatively clean because we have dispensed with the idea of a fixed pair $\mathfrak{v}_0,\mathfrak{v}_\infty$ of complementary parabolic subalgebras. In order to make contact with a classical version of the discussion, one need only choose a stereographic projection. It is clear however that T-transforms make perfect sense without any need for projections. As an example of this advantage at work, we use the connection formalism to provide a very quick proof of a standard identity: if (f,d,η) is isothermic,

[2]The sign of t is irrelevant: $K_t \in \operatorname{Aut}(\mathfrak{g})$ preserves $\mathfrak{v}_0,\mathfrak{v}_\infty$ and so, by Lemma 5.5, preserves M, M^*.

then

$$\mathcal{T}_s\mathcal{T}_t(f,\mathrm{d},\eta) = \mathcal{T}_s(f,\mathrm{d}^t,\eta) = (f,\mathrm{d}^t + s\eta,\eta) = (f,\mathrm{d}^{t+s},\eta) = \mathcal{T}_{t+s}(f,\mathrm{d},\eta). \quad (5.11)$$

T-transforms are therefore additive modulo the left action of G.

Darboux transforms

Notice that

$$\begin{aligned}(\mathrm{d} - t\eta_t)\mathcal{T}_t\tilde{f} &= \Phi_t \circ (\mathrm{d}^t - t\eta) \circ \Phi_t^{-1}\Phi_t \exp(F_\infty)\mathfrak{v}_\infty \\ &= \Phi_t \exp(F_\infty) \cdot (\mathrm{d}\mathfrak{v}_\infty + [\mathrm{d}F_\infty, \mathfrak{v}_\infty]) \subset \Omega^1_\Sigma \otimes \mathcal{T}_t\tilde{f}.\end{aligned} \quad (5.12)$$

This is an example of the *Darboux transform.*

Definition 5.9
Let (f,d,η) be isothermic, fix $t \in \mathbb{R}^\times$ and define $\mathrm{d}^t := \mathrm{d} + t\eta$. A Darboux transform *of f is a local d^t-parallel section $\hat{f}$ of the fibre bundle $\Sigma \times M^*$ such that $(f,\hat{f}) \in Z$. That is $\mathrm{d}^t_X\hat{f} \subset \hat{f}$, $\forall X \in \Gamma T\Sigma$. We write $\hat{f} = \mathcal{D}_t f$.*

Since $\mathrm{d}^t = \Phi_t^{-1} \circ \mathrm{d} \circ \Phi_t$ we can also define a Darboux transform of f by fixing an initial condition $o \in \Sigma$ and a fixed complement $\mathfrak{q}$ to $f(o)$, then setting $\mathcal{D}_t f := \Phi_t^{-1}\Phi_t(o) \cdot \mathfrak{q}$. Locally this is complementary to f and so there exists a unique Darboux transform through every fixed complementary parabolic subalgebra to $f(o)$. The existence of Darboux transforms cannot be deduced anything other than locally because of the set of complementary parabolic subalgebras to each $f(s)$, $s \in \Sigma$, although dense, is only open in M^*.

The Darboux transform was constructed by Darboux [24] as the transform from one enveloping surface of a Ribaucour sphere congruence in $\mathbb{R}^3$ (cf. Section 2.5) to the other. A modern account of this approach can be found in [15]. The three transforms (Christoffel, T- and Darboux) come together in a formula due to Bianchi [3] in $\mathbb{R}^3$, for (5.12) now reads

$$\mathcal{T}_t\tilde{f} = \mathcal{D}_{-t}\mathcal{T}_t f. \quad (5.13)$$

If the above expression is not enough to convince that we are really dealing with Darboux transforms, then the reader is invited to choose a stereo-projection and compare with the conformal n-sphere story in [11]. The above formula should also convince the reader that Darboux transforms are indeed isothermic, even though we have not exhibited a closed 1-form. This omission will remedied shortly.

We conclude this section by considering the interaction of general Darboux and T-transforms. Multiple applications of (5.11,5.13) tell us that up to the left action of G we have

$$\begin{aligned}\mathcal{D}_s\mathcal{T}_t f = \mathcal{D}_s\mathcal{T}_{-s}\mathcal{T}_{s+t}f = \mathcal{T}_{-s}\widetilde{\mathcal{T}_{s+t}f} = \mathcal{T}_{-s}\mathcal{T}_{s+t}\mathcal{D}_{s+t}\mathcal{T}_{-s-t}\mathcal{T}_{s+t}f \\ = \mathcal{T}_t\mathcal{D}_{s+t}f.\end{aligned} \tag{5.14}$$

(5.13) can be then be viewed as a limit of (5.14) as $s \to -t$. This idea will be explored further when we discuss the spectral deformation of a Darboux pair (5.19).

5.6 Bianchi Permutability of Darboux Transforms

A crucial part of the theory of Darboux transforms is the theorem of permutability: given two Darboux transforms f_1, f_2 of an isothermic surface f there generically exists a fourth isothermic surface f_{12} which is simultaneously a Darboux transform of f_1, f_2, and moreover is constructed via solely algebraic means. This statement remains true for isothermic submanifolds in *self-dual* symmetric R-spaces. In the non-self-dual case the problem is currently open, although permutability-type statements have been obtained in certain examples (see Chapter 6).

Proposition 5.10
Let (f,η) be an isothermic submanifold of a self-dual symmetric R-space M and let $f_1 = \mathcal{D}_{s_1}f$, $f_2 = \mathcal{D}_{s_2}f$ $(s_1 \neq s_2)$ be two Darboux transforms for which (f_1, f_2) are complementary. For $u \in \mathbb{R}$, define $r_u \in \mathrm{Aut}(\underline{\mathfrak{g}})$ by

$$r_u = \begin{cases} u & \text{on } f_1^{\perp}, \\ 1 & \text{on } f_1 \cap f_2, \cdot \\ u^{-1} & \text{on } f_2^{\perp}. \end{cases}$$

Then the family of connections

$$\tilde{\mathrm{d}}^t := r_{\frac{t-s_1}{t-s_2}} \circ \mathrm{d}^t \circ r^{-1}_{\frac{t-s_1}{t-s_2}}$$

can be written $\tilde{\mathrm{d}}^t = \tilde{\mathrm{d}} + t\eta$ where $\tilde{\mathrm{d}} = r_{s_1/s_2} \circ \mathrm{d} \circ r^{-1}_{s_1/s_2}$.

Self-duality is required since both f_1, f_2 are isothermic in the dual space M^* and so are *never* complementary for a non-self-dual M. The complementarity of (f_1, f_2) is generic in the sense of Proposition 1.13: the set of parabolic subalgebras complementary to f_1 is a dense open subset of M. Our theorem is only local, for if f_1, f_2 are complementary at a point $o \in \Sigma$, we only know that they are complementary in an open set around o. This resonates both with Burstall's discussion of Bianchi permutability of Darboux transforms in the conformal n-sphere [11], and with our discussion of permutability in Section 3.7.

Proof $\tilde{\mathrm{d}}^t$ is rational in t with poles possibly at s_1, s_2, ∞. Write $\mathrm{d}^{s_1} = D + \gamma_1 + \gamma_2$ where D is the canonical connection along $(f_1, f_2) : \Sigma \to Z$ and $\gamma_i \in \Omega^1_\Sigma \otimes f_i^\perp$. Since $f_1^\perp$ is d^{s_1}-parallel, we clearly have $\gamma_2 = 0$ and so

$$\mathrm{d}^t = D + \gamma_1 + (t - s_1)\eta.$$
$$\therefore \tilde{\mathrm{d}}^t = D = \frac{t - s_1}{t - s_2}\gamma_1 + (t - s_1) r_{\frac{t-s_1}{t-s_2}} \eta,$$

which is analytic at s_1. Similarly $\mathrm{d}^{s_2} f_2^\perp = 0 \Rightarrow \tilde{\mathrm{d}}^t$ is analytic at s_2. We can therefore express $\tilde{\mathrm{d}}^t$ as a power series in t about 0. However

$$\frac{1}{t}\tilde{\mathrm{d}}^t = r_{\frac{1-t^{-1}s_1}{1-t^{-1}s_2}} \circ (t^{-1}\mathrm{d} + \eta) \circ r^{-1}_{\frac{1-t^{-1}s_1}{1-t^{-1}s_2}} \xrightarrow[t\to\infty]{} = \eta$$

and so $\tilde{\mathrm{d}}^t = \tilde{\mathrm{d}} + t\eta$ where $\tilde{\mathrm{d}} = r_{s_1/s_2} \circ \mathrm{d} \circ r^{-1}_{s_1/s_2}$.

∎

Since $\tilde{\mathrm{d}}^t$ is flat (being a gauge transform of d^t) it is clear that $\tilde{\mathrm{d}}\eta = 0$ and so $r_{s_2/s_1}\eta$ is a closed 1-form with values in $r_{s_2/s_1} f^\perp$. $r_{s_2/s_1} f$ is therefore isothermic. Furthermore writing $\eta = a + b + c$ for the decomposition according to (f_1, f_2) we see that

$$\mathrm{d} + s_2 r_{s_2/s_1}\eta = \mathrm{d}^{s_1} + s_2 r_{s_2/s_1}\eta - s_1\eta = \mathrm{d}^{s_1} + (s_2^2/s_1 - s_1)a + (s_2 - s_1)b$$

has no $f_2^\perp$ component and so f_1 is $(\mathrm{d} + s_2 r_{s_2/s_1}\eta)$-parallel. Since r preserves f_1 we have that $(r_{s_2/s_1} f, f_1)$ are complementary and so $f_1 = \mathcal{D}_{s_2} r_{s_2/s_1} f$ which, by self-inversion gives $r_{s_2/s_1} f = \mathcal{D}_{s_2} f_1$. Similarly $r_{s_2/s_1} f = \mathcal{D}_{s_1} f_2$ and we have Bianchi permutability as in figure 5-1.

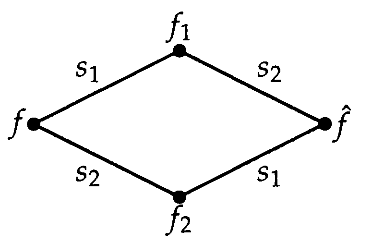

Figure 5-1: Bianchi permutability of Darboux transforms in self-dual G/P

We can get a general permutability-type statement for non-self-dual M as follows. Let f be isothermic, $f_1 = \mathcal{D}_{s_1} f$ and f^2 a d^{s_2}-parallel section of M: an *anti-Darboux transform* $f^2 = \mathcal{D}^{s_2} f$. We now generically have f_1, f^2 complementary and can build r_u as above. The analysis goes through unchanged, except that now $r_{s_2/s_1} f = \mathcal{D}_{s_2} f_1 = \mathcal{D}^{s_1} f^2$. As we shall see in Chapter 6, in certain non-self-dual M there are natural ways of building an anti-Darboux transform $f^2 = \mathcal{D}^{s_2} f$ via a duality map identifying a subset of the isothermic submanifolds of M with a subset of those of M^*. These yield permutability-type statements which at least involve two Darboux transforms.

5.7 Curved Flats

Recall that an immersion $\sigma = (f, \hat{f}) : \Sigma \to Z$ is a curved flat (Definition 1.7) iff the image $\mathcal{N}(T_s\Sigma)$ of each tangent space is an Abelian subalgebra iff (Proposition 1.8) the canonical connection $\mathcal{D}$ is flat. The zero-curvature condition (1.13) for d now reads

$$R^{\mathcal{D}} + [\beta \wedge \hat{\beta}] = 0 = \mathrm{d}^{\mathcal{D}} \beta = \mathrm{d}^{\mathcal{D}} \hat{\beta} \tag{5.15}$$

since $\mathcal{N} = \beta + \hat{\beta}$ and the nilradicals $f^{\perp}, \hat{f}^{\perp}$ are Abelian. It follows that σ is a curved flat iff $[\beta \wedge \hat{\beta}] = 0$.

Proposition 5.11
Let (f, η) be isothermic. If $\hat{f} = \mathcal{D}_t f$ then $\hat{\beta} = -t\eta$.

Proof A Darboux pair of isothermic submanifolds are always complementary and so $\mathcal{D}, \beta, \hat{\beta}$ are well-defined with respect to $(f, \hat{f})$. Now

$$\mathrm{d}^t \hat{f} = \mathcal{D}\hat{f} + \beta\hat{f} + (\hat{\beta} + t\eta) \cdot \hat{f} \subset \Omega^1_\Sigma \otimes \hat{f} \iff (\hat{\beta} + t\eta) \cdot \hat{f} \subset \Omega^1_\Sigma \otimes \hat{f}$$

which, since parabolic subalgebras are self-normalising (Lemma 1.12), implies $\hat{\beta} + t\eta \in \Omega^1_\Sigma \otimes \hat{f}$. However $\hat{\beta} + t\eta$ is $f^{\perp}$-valued and $f^{\perp} \cap \hat{f} = \{0\}$, thus $\hat{\beta} + t\eta = 0$.

■

The following theorem is a generalisation of the results in [15].

Theorem 5.12
Darboux pairs of isothermic submanifolds correspond to curved flats in the symmetric space Z.

Proof Let (f,η) be isothermic and $\hat{f} = \mathfrak{D}_t f$. Splitting flat d in the usual way we see that $\hat{\beta} = -t\eta$ and so $\mathrm{d}\hat{\beta} = 0$. However

$$\begin{aligned} \mathrm{d}\hat{\beta} &= (\mathcal{D} + \beta + \hat{\beta}) \cdot \hat{\beta} = \mathrm{d}^{\mathcal{D}}\hat{\beta} + [\beta \wedge \hat{\beta}] + [\hat{\beta} \wedge \hat{\beta}] \\ &= [\beta \wedge \hat{\beta}] \end{aligned} \tag{5.16}$$

by (5.15), and so $\mathcal{D}$ is flat.

Conversely, suppose $\mathcal{D}$ is flat. Then (5.15) and (5.16) tell us that $\beta, \hat{\beta}$ are closed, thus showing that $(f, \hat{\beta})$ and $(\hat{f}, \beta)$ are isothermic. Furthermore $\mathrm{d} - \hat{\beta} = \mathcal{D} + \beta$, for which $\hat{f}$ is clearly parallel and so $\hat{f} = \mathfrak{D}_{-1} f$ for $\eta := \hat{\beta}$.

■

The theorem answers a number of earlier questions. If $\hat{f} = \mathfrak{D}_t f$ we then *define* $\hat{\eta} = -\frac{1}{t}\beta$. $(\hat{f}, \hat{\eta})$ is clearly isothermic and our choice of scaling tells us that $f = \mathfrak{D}_t \hat{f}$ so that Darboux transforms are self-inverting. The issue of the maximal dimension of isothermic submanifolds can also be attacked: every isothermic f has a Darboux transform $\hat{f}$, the derivative of the resulting curved flat giving rise to a family of Abelian subalgebras of $f^{\perp} \oplus \hat{f}^{\perp}$. Indeed if we make the standard assumption (e.g. [9, 11, 56]) that each tangent space $\mathrm{d}(f, \hat{f})(T_s\Sigma)$ is conjugate to a fixed semisimple Abelian subalgebra then the maximal dimensions of such can often be calculated: this is how we arrive at the numbers 2, $\min(k, n-k)$ and $\lfloor n/2 \rfloor$ for our three examples. Since Z is not Riemannian, we cannot claim that maximal Abelian subalgebras are semisimple. Indeed the nilradicals $f^{\perp}, \hat{f}^{\perp}$ are Abelian subalgebras consisting entirely of nilpotent elements and have greater dimension, namely n, $k(n-k)$ and $\frac{1}{2}n(n-1)$. However we discount these possibilities on the grounds of interest, for $\mathcal{N} = -\eta - \hat{\eta} \Rightarrow$ one of $\eta, \hat{\eta}$ is zero and so Christoffel transforms are constant.

The Spectral Deformation

As seen in Section 4.2, curved flats come in natural 1-parameter families analogously to the construction of T-transforms. Let $\sigma : \Sigma \to G/K$ be a curved flat and let $\mathrm{d} = \mathcal{D} + \mathcal{N}$ be the decomposition of flat differentiation so that $\mathcal{D}$ is the canonical flat connection on $\Sigma \times \mathfrak{g}$ induced by the curved flat. The pencil of connections $\mathrm{d}_u := \mathcal{D} + u\mathcal{N}$ is flat since $\mathcal{N}$ is $\mathcal{D}$-closed. It is clear that σ is a curved flat with respect to d_u for any $u \in \mathbb{R}^{\times}$, since $\mathcal{D}_u = \mathcal{D}$ for all u: σ is constant for $\mathrm{d}_0 = \mathcal{D}$ and so *not* a curved flat. We call the family $(\sigma, \mathrm{d}_u)_{u \in \mathbb{R}^{\times}}$ the *spectral deformation* or *associated family* of σ.

In the same manner as for T-transforms we can find gauge transforms intertwining the connections d_u in order to build the spectral deformation in a more common fashion. $u\mathcal{N}$ satisfies the Maurer–Cartan equations for any $u \in \mathbb{R}$:

$$\mathrm{d}(u\mathcal{N}) + \frac{1}{2}[u\mathcal{N} \wedge u\mathcal{N}] = 0.$$

We can therefore locally integrate $S_u^{-1}\mathrm{d}S_u = (u-1)\mathcal{N}$ to get maps $S_u : \Sigma \to G$ satisfying

$$\mathrm{d}_u = S_u^{-1} \circ \mathrm{d} \circ S_u.$$

Fixing $S_1 = \mathrm{Id}$ and defining $\sigma_u := S_u \cdot \sigma$ allows us to identify

$$S_u : (\sigma, \mathrm{d}_u) \cong (\sigma_u, \mathrm{d}).$$

The family (σ_u) of curved flats is the more usual expression of the spectral deformation of σ, each σ_u being defined up to the left action of G. Supposing Σ to be simply connected, a choice of initial condition for all S_u at some $o \in \Sigma$ fixes a unique spectral deformation. When $G/K = Z$, observe that a choice of S_0 is equivalent to a choice of constant σ_0 which, being a constant pair of parabolic subalgebras, defines a stereographic projection.

Specialising to the symmetric space $Z \subset M \times M^*$ we demonstrate that the spectral deformation of a Darboux pair of isothermic submanifolds is given by the family of T-transforms of the Darboux pair. We have seen (Theorem 5.12) that there exists a unique Darboux pair $(f,\eta) = \mathcal{D}_1(\hat{f},\hat{\eta})$ such that $\sigma = (f,\hat{f})$ and $\mathcal{N} = -\eta - \hat{\eta}$. Fix a spectral deformation $\sigma_u = (f_u, \hat{f}_u)$.

Proposition 5.13
$f_u = \mathcal{T}_{1-u^2} f$, $\hat{f}_u = \mathcal{T}_{1-u^2}\hat{f}$ *for* $u \neq 0$.

Proof Let $K_s \in \mathrm{Aut}(\mathfrak{g})$ have eigenvalues $s, 1, s^{-1}$ on $f^\perp, f \cap \hat{f}, \hat{f}^\perp$ respectively and recall (Lemma 5.5) that M, M^* are K_s-stable for all $s \in \mathbb{R}^\times$. Observe that K gauges the spectral deformation connections with the family of flat connections defined by (f,η):

$$\mathrm{d}_u = K_u^{-1} \circ (\mathrm{d} + (1-u^2)\eta) \circ K_u = K_u^{-1} \circ \mathrm{d}^{1-u^2} \circ K_u.$$

Since $K_u \in \mathrm{Stab}(f,\hat{f})$ it follows that $(f, \mathrm{d}_u) = (K_u f, \mathrm{d}^{1-u^2}) = (f, \mathrm{d}^{1-u^2})$ and so $(f_u, \mathrm{d}) = (f, \mathrm{d}_u) = \mathcal{T}_{1-u^2} f$. Replacing K_u by $K_{u^{-1}}$ gives the corresponding result for $\hat{f}$. ∎

We may now apply (5.14) to see that

$$\hat{f}_u = \mathcal{D}_{u^2} f_u. \tag{5.17}$$

When $u = 0$ we no longer have a T-transform of f. According to (5.13) we can choose T-transforms

$$f' = \mathcal{T}_1 f, \qquad \hat{f}' = \mathcal{T}_1 \hat{f}$$

such that $(f', \hat{f}')$ are a Christoffel pair. (5.17) now reads as (5.13) with $t = -u^2$:

$$\hat{f}_u = \mathcal{T}_{-u^2}\hat{f}' = \mathcal{D}_{u^2}\mathcal{T}_{-u^2}f' = \mathcal{D}_{u^2}f_u. \tag{5.18}$$

This entire story can be viewed as taking the limit $\lim_{u\to 0} f_u$ of the spectral deformation in two different ways, since $K_u f = f$ for $u \neq 0$:

$$\begin{tikzcd} (f, \mathrm{d}_u) \arrow[r, equal, "K_u"] \arrow[d] \arrow[r, phantom, shift right=4ex, "\leftsquigarrow u \to 0 \rightsquigarrow"] & (f, \mathrm{d}^{1-u^2}) \arrow[d] \\ f_0 = (f, \mathcal{D}) & (f, \mathrm{d}^1) = \mathcal{T}_1 f. \end{tikzcd} \tag{5.19}$$

5.8 Dressing Curved Flats and $\mathfrak{m}$-flat Maps

In this section we abstract slightly the study of curved flats. We describe a method of transforming curved flats in any symmetric G-space for which $\mathfrak{g}^{\mathbb{C}}$ possesses parabolic subalgebras of height 1. This will be seen to correspond exactly to the dressing of $\mathfrak{p}$-flat maps by simple factors as described in Chapter 4. We will later return to isothermic submanifolds and apply the dressing transform to Darboux pairs in order to obtain another proof of the Bianchi permutability theorem for Darboux transforms in a self-dual symmetric R-space.

Let $Z = G/H$ be symmetric, $\sigma : \Sigma \to Z$ a curved flat and let τ be the symmetric involution along σ, i.e. with $\pm$-eigenspaces $\mathfrak{h} := \mathrm{stab}(\sigma)$, $\mathfrak{m} = \mathfrak{h}^{\perp}$ respectively: recall that $\mathfrak{h}, \mathfrak{m}$ are subbundles of $\underline{\mathfrak{g}} = \Sigma \times \mathfrak{g}$. Let the splitting of flat differentiation on $\underline{\mathfrak{g}}$ into the canonical connection and Maurer–Cartan form be $\mathrm{d} = \mathcal{D} + \mathcal{N}$ as usual. The spectral deformation of σ can then be viewed as the curved flat σ with respect to the family of flat connections $\mathrm{d}_u := \mathcal{D} + u\mathcal{N}$, $u \in \mathbb{R}^{\times}$. Fix $m \in \mathbb{R}^{\times}$ and suppose that $\mathfrak{w}$ is a d_m-parallel height 1 bundle of parabolic subalgebras of $\mathfrak{g}^{\mathbb{C}}$. Impose the generic condition that $\hat{\mathfrak{w}} := \tau\mathfrak{w}$ is complementary to $\mathfrak{w}$ and observe that this forces the canonical element $\xi_{\mathfrak{w}}$ of $(\mathfrak{w}, \hat{\mathfrak{w}})$ to live in $\mathfrak{m}$. Define a map r_u analogously to Proposition

5.10 such that

$$r_u = \begin{cases} u & \text{on } \mathfrak{w}^\perp, \\ 1 & \text{on } \mathfrak{w} \cap \hat{\mathfrak{w}}, \\ u^{-1} & \text{on } \hat{\mathfrak{w}}^\perp. \end{cases} \tag{5.20}$$

Observe that $\tau \circ r_u \circ \tau = r_{u^{-1}}$ and so in particular

$$r_{-1} \circ \mathrm{Proj}_{\mathfrak{m}} = \frac{1}{2}(r_{-1} - r_{-1} \circ \tau) = \frac{1}{2}(r_{-1} - \tau \circ r_{-1}) = \mathrm{Proj}_{\mathfrak{m}} \circ r_{-1}. \tag{5.21}$$

It follows that $\mathfrak{m}$ (similarly $\mathfrak{h}$) is r_{-1}-stable.

Introduce a second family of connections

$$\tilde{\mathrm{d}}_t := \mathcal{D} + t r_{-1} \mathcal{N}, \quad t \in \mathbb{R}.$$

Since $r_{-1}\mathcal{N} \in \Omega^1_\Sigma \otimes \mathfrak{m}$ it follows that $\tilde{\mathrm{d}}_t|_{\mathfrak{h}} = \mathcal{D}$ and so $(\sigma, \tilde{\mathrm{d}}_t)$ is a curved flat iff we can show that $\tilde{\mathrm{d}}_t$ is flat. The crucial observation is the following theorem—almost a carbon copy of Proposition 5.10 adapted to the curved flat connections instead of d^t.

Theorem 5.14
If $t \neq \pm m$, then $\tilde{\mathrm{d}}_t = r_u \circ \mathrm{d}_t \circ r_u^{-1}$ where $u = \frac{m-t}{m+t}$. Hence $\tilde{\mathrm{d}}_t$ is flat.

Proof Suppose $\tilde{\mathrm{d}}_t = r_u \circ \mathrm{d}_t \circ r_u^{-1}$ as in the theorem. $\tilde{\mathrm{d}}_t$ is rational in t with poles possibly at $t = \pm m, \infty$. Write $\mathrm{d}_m = D_{\mathfrak{w}} + \gamma + \hat{\gamma}$ where γ takes values in $\hat{\mathfrak{w}}^\perp$, etc. Since $\mathrm{d}_m \mathfrak{w} \subset \Omega^1_\Sigma \otimes \mathfrak{w}$, it is immediate that $\gamma \equiv 0$. Now $\mathrm{d}_t = \mathrm{d}_m + (t-m)\mathcal{N}$ and so

$$\tilde{\mathrm{d}}_t = r_u \circ \mathrm{d}_m \circ r_u^{-1} + (t-m) r_u \mathcal{N} = D_{\mathfrak{w}} + u\hat{\gamma} + (t-m) r_u \cdot \mathcal{N}.$$

The eigenvalues of r_u are $\frac{m-t}{m+t}, \frac{m+t}{m-t}, 1$, so $\tilde{\mathrm{d}}_t$ is analytic around $t = m$. By (5.20) we have that $\tilde{\mathrm{d}}_{-t} = \tau \circ \tilde{\mathrm{d}}_t \circ \tau$ and so $\tilde{\mathrm{d}}_t$ is also analytic near $t = -m$. However $\tilde{\mathrm{d}}_0 = r_1 \circ \mathcal{D} \circ r_1^{-1} = \mathcal{D}$ and

$$\lim_{t \to \infty} \frac{1}{t} \tilde{\mathrm{d}}_t = \lim_{t \to \infty} \left(\frac{1}{t} r_{u(t)} \circ \mathcal{D} \circ r_{u(t)}^{-1} + r_{u(t)} \mathcal{N} \right) = r_{-1} \mathcal{N},$$

since $u(t) \to -1$. We therefore have $\tilde{\mathrm{d}}_t = \mathcal{D} + t\, r_{-1} \mathcal{N}$. Since $\tilde{\mathrm{d}}_t$ is a gauge of the flat connection d_t it follows that $\tilde{\mathrm{d}}_t$ is flat.

∎

In particular we have shown that $(\sigma, \tilde{\mathrm{d}}_1) \cong (r_s \sigma, \mathrm{d})$ is a curved flat for $s = \frac{m+1}{m-1}$. Indeed we have actually transformed the entire associated family of σ for it is clear that the spectral deformation of $(\sigma, \tilde{\mathrm{d}}_1)$ is $(\sigma, \tilde{\mathrm{d}}_t)$. The following corollary is immediate.

Corollary 5.15
Let $(r_s\sigma, \hat{\mathrm{d}}_t := \hat{\mathcal{D}} + t\hat{\mathcal{N}})$ be the spectral deformation of the curved flat $(r_s\sigma, \mathrm{d})$. Since $(\sigma, \tilde{\mathrm{d}}_t)$ is the spectral deformation of the same curved flat $(\sigma, \tilde{\mathrm{d}}_1) \cong (r_s\sigma, \mathrm{d})$, it follows that $\hat{\mathrm{d}}_t = r_s \circ \tilde{\mathrm{d}}_t \circ r_s^{-1}$. Furthermore we have $\hat{\mathcal{D}} = r_s \circ \mathcal{D} \circ r_s^{-1}$ and $\hat{\mathcal{N}} = r_{-s}\mathcal{N}$ and that the following diagram commutes:

$$\begin{tikzcd}
(\sigma, \tilde{\mathrm{d}}_1) \arrow[r] \arrow[d, "r_s"'] \arrow[dr, phantom, "\substack{\text{spectral}\\ \rightsquigarrow \\ \text{deformation}}"] & (\sigma, \tilde{\mathrm{d}}_t) \arrow[d, "r_s"] \\
(r_s\sigma, \mathrm{d}) \arrow[r] & (r_s\sigma, \hat{\mathrm{d}}_t)
\end{tikzcd}$$

Summary Given a curved flat (σ, d) where $\mathrm{d}_t = \mathcal{D} + t\mathcal{N} = S_t^{-1} \circ \mathrm{d} \circ S_t$ is the spectral deformation of d, fix $m \in \mathbb{R}$ and a choice of d_m-parallel bundle of height 1 parabolic subalgebras $\mathfrak{w} \subset \underline{\mathfrak{g}}^{\mathbb{C}}$. This defines $r_u \in \operatorname{Aut} \underline{\mathfrak{g}}^{\mathbb{C}}$ and allows us to build two new families of flat connections:

$$\begin{aligned}
\tilde{\mathrm{d}}_t &:= \mathcal{D} + tr_{-1}\mathcal{N} = r_u \circ \mathrm{d}_t \circ r_u^{-1}, \\
\hat{\mathrm{d}}_t &:= \hat{\mathcal{D}} + t\hat{\mathcal{N}} = r_s \circ \tilde{\mathrm{d}}_t \circ r_s^{-1} = r_{su} \circ \mathrm{d}_t \circ r_{su}^{-1} = r_s \circ \mathcal{D} \circ r_s^{-1} + t\, r_{-s}\mathcal{N},
\end{aligned}$$

where $s = \frac{m+1}{m-1}$ and $u = \frac{m-t}{m+t}$. Viewing σ and $\hat{\sigma} := r_s\sigma$ with respect to these connections gives different families of curved flats:

Associated family $(\sigma, \mathrm{d}_t) = (\sigma_t = S_t \cdot \sigma, \mathrm{d})$
Dressing $(\sigma, \tilde{\mathrm{d}}_1) \leftrightsquigarrow (\hat{\sigma}, \mathrm{d})$
Associated dressed family $(\sigma, \tilde{\mathrm{d}}_t) \leftrightsquigarrow (\hat{\sigma}_t = S_t r_u^{-1} \cdot \sigma, \mathrm{d}) \leftrightsquigarrow (\hat{\sigma}, \hat{\mathrm{d}}_t)$.

For the sake of interest we now give an alternative proof that $r_s\sigma$ is a curved flat, which also gives us a different view of the connection $\tilde{\mathrm{d}}_1$.

Proposition 5.16
Let $\mathfrak{w}$ be a d_m-parallel bundle of height 1 parabolic subalgebras of $\mathfrak{g}^{\mathbb{C}}$ such that $\hat{\mathfrak{w}} = \tau\mathfrak{w}$ is complementary to $\mathfrak{w}$ and define r_u as in (5.20) and $s = \frac{m+1}{m-1}$. Then $r_{-s}^{-1} \circ \mathrm{d} \circ r_{-s} = \mathrm{d} - 2m\,\mathrm{ad}(\xi_{\mathfrak{w}})\mathcal{N}$.

Proof Write $\mathrm{d} = D + \alpha + \hat{\alpha}$ where $\alpha \in \Omega^1_\Sigma \otimes \mathfrak{w}^\perp$, etc. Thus

$$\mathrm{d}_m - D - \alpha = \hat{\alpha} - (1-m)\mathcal{N}.$$

The LHS preserves $\mathfrak{w}$ and so the projection of the RHS onto $\mathfrak{w}^\perp$ is zero. Similarly $\hat{\mathfrak{w}}$ is

d_{-m}-parallel and so $\alpha - (1+m)\mathcal{N}$ has no $\hat{\mathfrak{w}}^{\perp}$ component. Thus

$$\alpha = (1+m)\mathrm{Proj}_{\mathfrak{w}^{\perp}}\mathcal{N}, \qquad \hat{\alpha} = (1-m)\mathrm{Proj}_{\hat{\mathfrak{w}}^{\perp}}\mathcal{N}.$$

It follows that

$$\begin{aligned} r_{-s}^{-1} \circ \mathrm{d} \circ r_{-s} &= D - s^{-1}\alpha - s\hat{\alpha} = \mathrm{d} + (1+s^{-1})\alpha + (1+s)\hat{\alpha} \\ &= \mathrm{d} - (1+s^{-1})(1+m)\mathrm{Proj}_{\mathfrak{w}^{\perp}}\mathcal{N} - (1+s)(1-m)\mathrm{Proj}_{\hat{\mathfrak{w}}^{\perp}}\mathcal{N} \\ &= \mathrm{d} - 2m\,\mathrm{ad}(\xi_{\mathfrak{w}})\mathcal{N}. \end{aligned}$$

■

$r_s\sigma = r_{-s}\sigma$ is a curved flat iff $(\sigma, r_{-s}^{-1} \circ \mathrm{d} \circ r_{-s})$ is a curved flat. However $\mathrm{ad}(\xi_{\mathfrak{w}})\mathcal{N}$ takes values in $\mathfrak{h}$ (since $\xi_{\mathfrak{w}} \in \mathfrak{m}$) and so the only part of $r_{-s}^{-1} \circ \mathrm{d} \circ r_{-s}$ with values in $\mathfrak{m}$ is $\mathcal{N}$. The canonical connection along σ with respect to $r_{-s}^{-1} \circ \mathrm{d} \circ r_{-s}$ is therefore flat and so $r_s\sigma$ is a curved flat.

$\mathfrak{m}$-flat maps

We now demonstrate that the above dressing action on curved flats exactly corresponds to the dressing of $\mathfrak{p}$-flat maps described in Chapter 4. In Chapter 4 a $\mathfrak{p}$-flat map took values in the $-ve$-eigenspace of a fixed (d-parallel) symmetric involution. Since this chapter is formulated without the choice of a preferred base point we will consequently define $\mathfrak{m}$-flat maps.

Definition 5.17
Let $\sigma : \Sigma \to Z$ be a curved flat in a symmetric space (Z, τ) where τ has $\pm$-eigenspaces $\mathfrak{h} = \mathrm{stab}(\sigma), \mathfrak{m} = \mathfrak{h}^{\perp}$. Let $\mathcal{D}$ be the canonical connection on the bundle $\underline{\mathfrak{g}}$ induced by σ. A map $\psi : \Sigma \to \mathfrak{m}$ is $\mathfrak{m}$-flat iff $[\mathcal{D}\psi \wedge \mathcal{D}\psi] = 0$.

If we fix a gauge-transform S_0 such that $\sigma_0 = S_0\sigma$ is the d-constant term in the spectral deformation of σ, then $S_0\psi : \Sigma \to \mathfrak{p} := S_0\mathfrak{m}$ is a $\mathfrak{p}$-flat map in the sense of Section 4.2. The two definitions are therefore gauge-equivalent. Because of this we can invoke the entire discussion of Chapter 4 by systematically replacing d with $\mathcal{D}$ and $\mathfrak{p}$-flat with $\mathfrak{m}$-flat. Explicitly this works as follows. Given an $\mathfrak{m}$-flat map ψ, the 1-parameter family of $\mathfrak{m}$-flat maps $\psi_t := t\psi$ satisfy the Maurer–Cartan equations

$$\mathcal{D}(\mathcal{D}\psi_t) + \frac{1}{2}[\mathcal{D}\psi_t \wedge \mathcal{D}\psi_t] = 0$$

for all $t \in \mathbb{C}$. We therefore integrate $\mathcal{D}(t\psi) = t\mathcal{N} = P_t^{-1}\mathrm{d}P_t$ to get a holomorphic map

$$P_t : \Sigma \to G^{\mathbb{C}}, \quad P_0 = \mathrm{Id}, \ \tau P_t = P_{-t}$$

unique (if Σ simply connected) up to left translations by $\mathcal{D}$-parallel maps into $G^{\mathbb{C}}$: an *extended flat frame* of ψ. Observe that P_t gauges the connections d_t and $\mathcal{D}$ and so we can choose initial conditions such that $P_t = S_0^{-1}S_t$, $S_t = P_1^{-1}P_t$. It is clear that $\Phi_t := P_1^{-1}P_tP_1$ is an extended flat frame of the $\mathfrak{p}$-flat map $P_1^{-1}\psi$ in the sense of Section 4.2. Lemma 4.3 and the Sym formula (4.3) both hold (simply conjugate by P_1) so that a holomorphic map $\hat{P}_t : \Sigma \times \mathbb{C} \to G^{\mathbb{C}}$ is an extended flat frame iff $\hat{P}_t^{-1}\mathcal{D}P_t$ has a simple pole at ∞ and

$$\psi = \left.\frac{\partial P_t}{\partial t}\right|_{t=0} + \psi_0$$

up to the addition of a $\mathcal{D}$-parallel section $\psi_0 \in \Gamma\mathfrak{m}$.

Recall our earlier expressions for the associated family of the curved flat $\hat{\sigma} = r_s\sigma$:

$$\hat{\sigma}_t = (S_tr_{us}^{-1}\hat{\sigma}, \mathrm{d}) = (P_tr_u^{-1}\sigma, \mathcal{D}).$$

Since, for consistency, we must view everything with respect to the same connection ($\mathcal{D}$) and base (σ) it is clear that the extended flat frame of the associated family of $\hat{\sigma}$ is $\hat{P}_t = P_tr_u^{-1}$. Since $\hat{P}_t^{-1}\mathcal{D}\hat{P}_t = tr_{-1}\mathcal{N}$ has a simple pole at ∞ we can define a new $\mathfrak{m}$-flat map:

$$\hat{\psi} := \left.\frac{\partial \hat{P}_t}{\partial t}\right|_{t=0} = \left.\frac{\partial P_t}{\partial t}\right|_{t=0} + \left.\frac{\partial r_u^{-1}}{\partial t}\right|_{t=0} = \psi + \frac{2}{m}\xi_{\mathfrak{w},\hat{\mathfrak{w}}},$$

since $r_u = \exp\left(\ln \frac{m-t}{m+t}\xi_{\mathfrak{w},\hat{\mathfrak{w}}}\right)$. Unsurprisingly this is exactly the expression one expects after dressing ψ by a simple factor. Indeed we can generically recover the simple factor. Observe that the parabolic subalgebras

$$\mathfrak{q} := P_m\mathfrak{w}, \qquad \mathfrak{r} := P_{-m}\hat{\mathfrak{w}}$$

are $\mathcal{D}$-parallel and that $\tau\mathfrak{q} = \mathfrak{r}$. Supposing that $(\mathfrak{q}, \mathfrak{r})$ are complementary we can form the simple factor

$$p_{m,\mathfrak{q}}(t) := \exp\left(\ln \frac{m-t}{m+t}\xi_{\mathfrak{q},\mathfrak{r}}\right).$$

In accordance with Chapter 4, the dressing action of $p_{m,\mathfrak{q}}$ on the flat frame P_t and the $\mathfrak{m}$-flat map ψ are then

$$p_{m,\mathfrak{q}} \# P_t = p_{m,\mathfrak{q}}P_tp^{-1}_{m,P_m^{-1}\mathfrak{q}} = p_{m,\mathfrak{q}}P_tp^{-1}_{m,\mathfrak{w}} = p_{m,\mathfrak{q}}P_tr_u^{-1},$$

$$p_{m,\mathfrak{q}} \# \psi = \psi + \frac{2}{m} \xi_{\mathfrak{w},\hat{\mathfrak{w}}},$$

as required.

Even more is true, for the permutability theorem for dressing by simple factors (Proposition 4.5) may be translated to give a permutability theorem for the dressing of curved flats.

Proposition 5.18
Let σ be a curved flat with spectral deformation $\mathrm{d}_t = \mathcal{D} + t\mathcal{N}$. Let $\mathfrak{w}_1, \mathfrak{w}_2$ be height 1 bundles of parabolic subalgebras which are $\mathrm{d}_m, \mathrm{d}_n$-parallel respectively such that $m^2 \neq n^2$ and suppose that $\hat{\mathfrak{w}}_i := \tau\mathfrak{w}_i$ are complementary to $\mathfrak{w}_i$, $i = 1, 2$, so that we may define the maps

$$r_{u_1}(t) := \begin{cases} \frac{m-t}{m+t} & \text{on } \mathfrak{w}_1^{\perp}, \\ 1 & \text{on } \mathfrak{w}_1 \cap \hat{\mathfrak{w}}_1, , \\ \frac{m+t}{m-t} & \text{on } \hat{\mathfrak{w}}_1^{\perp}, \end{cases} \qquad r_{u_2}(t) := \begin{cases} \frac{n-t}{n+t} & \text{on } \mathfrak{w}_2^{\perp}, \\ 1 & \text{on } \mathfrak{w}_2 \cap \hat{\mathfrak{w}}_2, \\ \frac{n+t}{n-t} & \text{on } \hat{\mathfrak{w}}_2^{\perp}. \end{cases}$$

Write $s_i = u_i(-1)$ so that we have new curved flats $\sigma_i = r_{s_i}\sigma$. Define new bundles of parabolic subalgebras

$$\mathfrak{w}_1' = r_{s_2 u_2}(m)\mathfrak{w}_1, \quad \mathfrak{w}_2' = r_{s_1 u_1}(n)\mathfrak{w}_2,$$
$$\hat{\mathfrak{w}}_1' = r_{s_2 u_2}(-m)\mathfrak{w}_1, \quad \hat{\mathfrak{w}}_2' = r_{s_1 u_1}(-n)\hat{\mathfrak{w}}_2,$$

and observe that $\hat{\mathfrak{w}}_i = \tau\mathfrak{w}_i$ where τ is the symmetric involution with respect to $r_{s_i}\sigma$. Assume that $(\mathfrak{w}_i', \hat{\mathfrak{w}}_i')$ are complementary for each i so that we can form r_{u_i}' in the usual way. Then (generically) we have

$$r_{s_1}' r_{s_2} = r_{s_2}' r_{s_1}$$

and $\sigma_{12} := r_{s_2}'\sigma_1 = r_{s_1}'\sigma_2$ is a simultaneous dressing transform of σ_1, σ_2.

By 'generic' we mean that several further pairs of parabolic subalgebras are required to be complementary. The proof is somewhat long-winded, but consists of little more than translating Proposition 4.5 to a statement about 'dressed' simple factors together with a number of gauge transforms.

Proof Suppose r_{u_1}, r_{u_2} are given as in the proposition. Define bundles of parabolic subalgebras

$$W_1' = r_{u_2}(m)\mathfrak{w}_1, \quad W_2' = r_{u_1}(n)\mathfrak{w}_2,$$

$$\hat{W}_1' = r_{u_2}(-m)\mathfrak{w}_1, \quad \hat{W}_2' = r_{u_1}(-n)\hat{\mathfrak{w}}_2,$$

and define R_{u_1}, R_{u_2} with respect to $(W_1', \hat{W}_1')$, etc. in the usual manner.[3]

Let P_t frame the associated family of σ so that $P_t^{-1}\mathcal{D}P_t = t\mathcal{N}$. By the above discussion we (generically) have simple factors $p_1 = p_{m,P_m\mathfrak{w}_1,P_{-m}\hat{\mathfrak{w}}_1}, p_2 = p_{n,P_n\mathfrak{w}_2,P_{-n}\hat{\mathfrak{w}}_2}$. Following Proposition 4.5 we define

$$\begin{aligned}\mathfrak{q}_1' &= p_2(m)P_m\mathfrak{w}_1, \quad \mathfrak{q}_2' = p_1(n)P_n\mathfrak{w}_2,\\ \mathfrak{r}_1' &= p_2(-m)P_{-m}\hat{\mathfrak{w}}_1, \quad \mathfrak{r}_2' = p_1(-n)P_{-n}\hat{\mathfrak{w}}_2\end{aligned}$$

and observe that the simple factors $p_1' = p_{m,\mathfrak{q}_1',\mathfrak{r}_1'}, p_2' = p_{n,\mathfrak{q}_2',\mathfrak{r}_2'}$ satisfy $p_1'p_2 = p_2'p_1$.

Consider the dressing action of p_2' on $\tilde{P}_{1,t} := p_1 \# P_t$:

$$p_2' \# \tilde{P}_{1,t} = p_2'\tilde{P}_{1,t}p^{-1}_{n,\tilde{P}_{1,n}^{-1}\mathfrak{q}_2',\tilde{P}_{1,-n}^{-1}\mathfrak{r}_2'}.$$

Now

$$\tilde{P}_{1,n}^{-1}\mathfrak{q}_2' = r_{u_1}(n)\tilde{P}_n^{-1}p_1(n)^{-1}p_1(n)\tilde{P}_n\mathfrak{w}_2 = r_{u_1}(n)\mathfrak{w}_2 = W_2',$$

etc. from which it is clear that $R_{u_1}r_{u_2} = R_{u_2}r_{u_1}$ is the 'dressed' equivalent of the relation $p_1'p_2 = p_2'p_1$.

Define $\mathfrak{w}_i', \hat{\mathfrak{w}}_i'$ as in the proposition. Observe from Corollary 5.15 that the spectral deformation of d with respect to $\sigma_1 = r_{s_1}\sigma$ is $\hat{\mathrm{d}}_{1,t} = r_{s_1u_1} \circ \mathrm{d}_t \circ r_{s_1u_1}^{-1}$. It follows that

$$\hat{\mathrm{d}}_{1,n}\mathfrak{w}_2' = r_{s_1u_1}(n) \circ \mathrm{d}_n\mathfrak{w}_2 = 0,$$

etc. We may therefore define r_{u_i}' in the usual way with respect to $(\mathfrak{w}_i', \hat{\mathfrak{w}}_i')$ and observe that $r_{s_2}'\sigma_1$ is a dressing transform of σ_1 (similarly $r_{s_1}'\sigma_2$). To finish, note that $R_{u_1} = r_{s_2}^{-1}r_{u_1}'r_{s_2}$, etc., which, by setting $t = 1$, yields

$$r_{s_2}'r_{s_1} = r_{s_1}'r_{s_2}.$$

Hence $\sigma_{12} := r_{s_2}'\sigma_1 = r_{s_1}'\sigma_2$ is well-defined and a simultaneous dressing transform of σ_1, σ_2.

∎

[3] $(W_i', \hat{W}_i')$ are complementary by assumption in the proposition.

5.9 Dressing Darboux Pairs and the Bianchi Cube

For the remainder of this chapter we return to the symmetric space Z of pairs of complementary parabolic subalgebras in $M \times M^*$. Let $\sigma = (f, \hat{f})$ be a Darboux pair. Fix m and a d_m-parallel bundle of parabolic subalgebras $\mathfrak{w}$ of $\mathfrak{g}^{\mathbb{C}}$ where we impose the generic condition that $\hat{\mathfrak{w}} := \tau\mathfrak{w}$ is complementary to $\mathfrak{w}$. There is no need to assume that the conjugacy class of $\mathfrak{w}$ coincides with that of f or $\hat{f}$, but notice that if it does then this *requires* M to be self-dual, since $\tau \in \mathrm{Aut}(\overline{\mathfrak{g}})$ preserves $f, \hat{f}$ (Lemma 5.5). In such a case we note immediately that the symmetric involution on Z with respect to $(\mathfrak{w}, \hat{\mathfrak{w}})$ is $\tau|_{\mathfrak{w}} = r_{-1}$.

For the present let us return to the general situation where we make no assumption on $\mathfrak{w}$.

Lemma 5.19
We have a symmetric set-up: if $\zeta_f, \zeta_{\mathfrak{w}}$ are the canonical elements of $(f, \hat{f})$, $(\mathfrak{w}, \hat{\mathfrak{w}})$ respectively, then $\zeta_{\mathfrak{w}} \in \mathfrak{m} = f^{\perp} \oplus \hat{f}^{\perp}$ and $\zeta_f \in \mathfrak{w}^{\perp} \oplus \hat{\mathfrak{w}}^{\perp}$. It follows that $r_{-1}f = \hat{f}$.

Proof The first claim is obvious from $\tau\mathfrak{w} = \hat{\mathfrak{w}}$. For the second, $[\tau, r_{-1}] = 0$ implies (5.21) that $r_{-1}\mathfrak{h} = \mathfrak{h}$ and so $r_{-1}\zeta_f \in \mathfrak{z}(\mathfrak{h})$, the centre of $\mathfrak{h}$. It is enough to restrict to simple $\mathfrak{g}^{\mathbb{C}}$ where we have[4] $\dim \mathfrak{z}(\mathfrak{h}) = 1$, since, by the final remarks of Section 1.3, the complementary pairs $(f, \hat{f})$ and $(\mathfrak{w}, \hat{\mathfrak{w}})$ restrict to (possibly trivial) complementary pairs in any simple ideal and so τ and r_{-1} preserve a decomposition of $\mathfrak{g}^{\mathbb{C}}$ into simple ideals. It follows that ζ_f is an eigenvector of r_{-1}, necessarily with eigenvalue ± 1. Hence $\zeta_f \in \mathfrak{w} \cap \hat{\mathfrak{w}}$ or $\mathfrak{w}^{\perp} \oplus \hat{\mathfrak{w}}^{\perp}$. However $\tau = \mathrm{Id} - 2\,\mathrm{ad}^2 \zeta_f : \mathfrak{w}^{\perp} \mapsto \hat{\mathfrak{w}}^{\perp}$ and so $\zeta_f \in \mathfrak{w} \cap \hat{\mathfrak{w}}$ is a contradiction. Finally it is immediate from $r_{-1}\zeta_f = -\zeta_f$ that $r_{-1}f = \hat{f}$. ∎

Considering the connection $\mathrm{d} = \hat{\mathrm{d}}_1 = r_s \circ \mathcal{D} \circ r_s^{-1} + r_{-s}\mathcal{N}$ as described in the previous section we see that $r_{-s}\mathcal{N}$ is the Maurer–Cartan form of the dressed curved flat $r_s\sigma$. Since $r_{-1}f = \hat{f}$ by Lemma 5.19 it is clear that $r_s f, r_s\hat{f}$ are isothermic with closed 1-forms $r_{-s}\hat{\eta}, r_{-s}\eta$ respectively. Indeed if $\hat{f} = \mathcal{D}_1 f$, we see that

$$
\begin{aligned}
(\mathrm{d} + r_{-s}\hat{\eta})r_s\hat{f} &= r_s(\mathcal{D} + r_{-1}\mathcal{N} + r_{-1}\hat{\eta})\hat{f} = r_s(\mathcal{D} - r_{-1}\eta)\hat{f} \\
&= r_s(\mathcal{D}\hat{f} - r_{-1}(\eta f)) \subset \Omega^1_{\Sigma} \otimes r_s\hat{f},
\end{aligned}
\tag{5.22}
$$

and so $r_s\hat{f} = \mathcal{D}_1 r_s f$.

We are almost in a position to be able to state another theorem of permutability for Darboux transforms, for a Darboux transform $\tilde{f} = \mathcal{D}_{1-m^2} f$ is synonymous with a choice of d_m-parallel subbundle $\mathfrak{w}$ of $\Sigma \times M$ as the next theorem shows.

[4] Pg. 42, Remark(iii) of [18]: if $f = \mathfrak{q}_I$ as in Section 1.3 then $\dim \mathfrak{z}(\mathfrak{h}) = |I|$.

Theorem 5.20
Let f be isothermic in a self-dual symmetric R-space M. Given two Darboux transforms $\hat{f} = \mathcal{D}_1 f$, $\tilde{f} = \mathcal{D}_p f$ ($p < 1$) of f, let $m^2 = 1 - p$ and define $\mathfrak{w} = K_m^{-1}\tilde{f}$ where K_m is the usual automorphism with eigenspaces $f^\perp, f \cap \hat{f}, \hat{f}^\perp$ (cf. pg. 107). Then $\mathfrak{w}$ is d_m-parallel. If we furthermore assume that $(\mathfrak{w}, \hat{\mathfrak{w}})$ are complementary[5] *then $\tilde{f} = r_s f$, where $s = \frac{m+1}{m-1}$. Conversely, if $\mathfrak{w}$ is a d_m-parallel subbundle of $\Sigma \times M$ such that $(\mathfrak{w}, \tau\mathfrak{w})$ are complementary then $\tilde{f} := K_m \mathfrak{w} = \mathcal{D}_{1-m^2} f$.*

It may not be obvious from the following proof why we must restrict to self-dual M. The requirement that $p < 1$ is no restriction, for given any two Darboux transforms $\mathcal{D}_a f, \mathcal{D}_b f$ one can scale η suitably so that these become $\mathcal{D}_1 f, \mathcal{D}_p f$ where $p < 1$. The only purpose of this is to force K_m to be a *real* automorphism whenever M is a real conjugacy class. Since K_m preserves $f, \hat{f}$ it follows (Lemma 5.5) that K_m preserves M, M^* and so $\mathfrak{w} \in M^*$. Since $r_{-1} \in \mathrm{Aut}(\underline{\mathfrak{g}})$ preserves $\mathfrak{w}$, a second application of Lemma 5.5 forces M to be self-dual. When M is complex the reality or otherwise of K_m is no issue and Lemma 5.5 automatically forces M to be self-dual.

Proof The claim that $\mathfrak{w}$ is d_m-parallel is straight from the argument of Proposition 5.13 since K_m gauges the two connections $\mathrm{d}_m, \mathrm{d}^{1-m^2}$. Let $\tilde{F}$ be a section of $\tilde{f}^\perp$ and define the corresponding section $v = K_m^{-1}\tilde{F}$ of $\mathfrak{w}^\perp$. Supposing complementarity of $(\mathfrak{w}, \hat{\mathfrak{w}})$ we let $\hat{v} = \tau v$ and observe that $v - \hat{v}$ is skew for τ. There therefore exist unique sections $F \in f^\perp$, $\hat{F} \in \hat{f}^\perp$ such that

$$v - \hat{v} = F + \hat{F}.$$

By Lemma 5.19 ($\xi_f \in \mathfrak{w}^\perp \oplus \hat{\mathfrak{w}}^\perp$) we see that $F - \hat{F} \in \mathfrak{w} \cap \hat{\mathfrak{w}}$ and so

$$F = \frac{1}{2}(v - \hat{v}) + \frac{1}{2}(F - \hat{F}) \tag{5.23}$$

is the unique decomposition of F into $\mathfrak{w}^\perp \oplus \hat{\mathfrak{w}}^\perp$ and $\mathfrak{w} \cap \hat{\mathfrak{w}}$ parts. We can now apply r_s:

$$\begin{aligned}
r_s F &= \frac{1}{2} s v - \frac{1}{2} s^{-1}\hat{v} + \frac{1}{2}(F - \hat{F}), \\
r_s F + \tau r_s F &= \frac{1}{2}(s - s^{-1})(v + \hat{v}) = \frac{2m}{m^2 - 1}(v + \hat{v}), \\
r_s F - \tau r_s F &= \frac{1}{2}(s + s^{-1})(v - \hat{v}) + F - \hat{F} = \frac{1}{2}((s + s^{-1} + 2)F + (s + s^{-1} - 2)\hat{F}), \\
&= \frac{2m}{m^2 - 1}(mF + m^{-1}\hat{F}).
\end{aligned}$$

[5] Equivalently $(\tilde{f}, \tau\tilde{f})$ are complementary.

Now $\tau \circ K_m = K_{-m} = K_m \circ \tau$, and so

$$K_m^{-1} r_s F = \frac{m}{m^2-1}(v + \hat{v} + F + \hat{F}) = \frac{2m}{m^2-1} v,$$
$$\implies r_s F = \frac{2m}{m^2-1} \tilde{F}.$$

Hence $\tilde{f}^{\perp} = r_s f^{\perp}$ and we have proved the first claim.

For the converse observe that $\tilde{f} := K_m \mathfrak{w}$ is clearly d^{1-m^2}-parallel since K_m gauges d^{1-m^2} and d_m. Thus, assuming complementarity of $(f, \tilde{f})$, we have $\tilde{f} = \mathcal{D}_{1-m^2} f$. The above calculation also shows that $\tilde{f} = r_s f$, as required.

∎

As advertised, the above theorem will result in another theorem on the permutability of Darboux transforms (Corollary 5.21), but we will delay this until we have used the above theorem to see what effect the dressing transform of curved flats has on $\mathfrak{m}$-flat maps in this context.

Suppose $\sigma = (f, \hat{f})$ is a Darboux pair. An $\mathfrak{m}$-flat map ψ obtained by integrating the Maurer–Cartan form $\mathcal{N}$ splits into two pieces $\psi = \psi_1 + \psi_2$ where $(\psi_1, \psi_2) : \Sigma \to f^{\perp} \times \hat{f}^{\perp}$. Clearly $\mathcal{D}\psi_1 = \hat{\beta}$, $\mathcal{D}\psi_2 = \beta$. Then

$$f' := (\exp(\psi_2) f, \mathcal{D}, \exp(\psi_2)\eta), \qquad \hat{f}' = (\exp(\psi_1)\hat{f}, \mathcal{D}, \exp(\psi_1)\hat{\eta})$$

are a Christoffel pair (Definition 5.7) with respect to $\mathcal{D}$ since $[\mathcal{D}\psi_2 \wedge \mathcal{D}\psi_1] = 0$. The choice of notation $f', \hat{f}'$ is justified since these are indeed the Christoffel pair given by blowing up the associated family as $u \to 0$ (5.19): since

$$\exp(-\psi_2) \circ \mathcal{D} \circ \exp(\psi_2) = \mathcal{D}\psi_2 + \mathcal{D} = \mathrm{d} + \eta,$$

we have

$$f' = (f, \mathrm{d} + \eta, \eta) = \mathcal{T}_1 f.$$

Since T-transforms are additive we have $f_u = \mathcal{T}_{-u^2} f'$ as in (5.18). The result for $\hat{f}'$ is similar.

It remains to see what dressing does to the Christoffel pair $\psi = (f', \hat{f}')$. This is straightforward, for by Theorem 5.20, $(r_s f, r_s \hat{f}) = (\mathcal{D}_{1-m^2} f, \mathcal{D}_{1-m^2} \hat{f})$ and so

$$\hat{\psi} = \mathcal{T}_1 \circ \mathcal{D}_{1-m^2} \circ \mathcal{T}_{-1} \psi = \mathcal{D}_{-m^2} \psi,$$

by (5.14). As summarised in figure 5-2 the dressing transform therefore acts by Darboux transforms on the Christoffel pair $(f', \hat{f}')$.

$$\begin{tikzcd}
\sigma = (f, \hat{f}) \arrow[r] \arrow[d, "\mathcal{D}_{1-m^2}"'] \arrow[r, phantom, shift right=6ex, "\text{blowup}"] & \psi = (f', \hat{f}') \arrow[d, "\mathcal{D}_{-m^2}"] \\
r_s \sigma \arrow[r] & \hat{\psi}
\end{tikzcd}$$

Figure 5-2: Dressing curved flats versus $\mathfrak{m}$-flat maps

Returning to the dressing of a Darboux pair, we use Theorem 5.20 to obtain a second proof of the permutability of Darboux transforms.

Corollary 5.21 (Bianchi permutability)
Let f be isothermic in a self-dual symmetric R-space M. Let $\hat{f} := \mathcal{D}_p f$ be a Darboux transform such that the involution on Z with respect to $(f, \hat{f})$ is τ. Suppose $\tilde{f} := \mathcal{D}_q f$ ($p \neq q$) is a second Darboux transform such that $(\tilde{f}, \tau\tilde{f})$ are complementary. Then there exists a common Darboux transform $\mathcal{D}_q \hat{f} = \mathcal{D}_p \tilde{f}$.

Remarks: We now have two distinct theorems (cf. Section 5.6) of the Bianchi permutability of Darboux transforms in a self-dual symmetric R-space, both of which have the fourth submanifold constructed by solely algebraic means. When both theorems apply we do *not* claim that the fourth submanifolds constructed are the same in both cases. Both proofs contrast radically with the discussions of permutability in $\mathbb{R}^n$ given by Burstall [11] and Schief [52], where the result relies on the four transforms being suitably concircular. Moreover we make no claim about the uniqueness of a fourth Darboux transform.

Proof By writing $(f, \tilde{f}) = g \cdot (f, \hat{f})$ for some $g : \Sigma \to G$ it is easy to see that the assumption that $(\tilde{f}, \tau\tilde{f})$ be complementary is equivalent to the complementarity of $(\hat{f}, \tau_{f,\tilde{f}}\hat{f})$. We can therefore assume without loss of generality that $q/p < 1$ and rescale $\eta, \hat{\eta}$ so that $\hat{f} := \mathcal{D}_1 f$ and $\tilde{f} = \mathcal{D}_{1-m^2} f$ where $m = \sqrt{1 - q/p} \in \mathbb{R}^\times$. Then Theorem 5.20 says that

$$\tilde{f} = r_s f \text{ where } s = \frac{m+1}{m-1}, \quad \mathfrak{w} := K_m^{-1} \tilde{f}.$$

It remains to show that $r_s \hat{f} = \mathcal{D}_{1-m^2} \hat{f} = \mathcal{D}_1 \tilde{f}$. The second of these is just (5.22) with η' replacing $\hat{\eta}$. For the first, rearrange (5.23) to get $\hat{F} = \frac{1}{2}(v - \hat{v} - F + \hat{F})$ and apply r_s as in the proof to see that

$$r_s f = K_{m^2} r_s \hat{f}.$$

But $(\mathrm{d} + (1 - m^2)\eta) \circ K_{m^2} = K_{m^2} \circ (\mathrm{d} + (1 - m^2)\hat{\eta})$ and so

$$(\mathrm{d} + (1 - m^2)\hat{\eta})r_s\hat{f} = K_{m^2}^{-1}(\mathrm{d} + (1 - m^2)\eta)r_s f \subset \Omega^1_\Sigma \otimes r_s\hat{f}$$

since $r_s f = \mathcal{D}_{1-m^2} f$. Unscaling $\eta, \hat{\eta}$ gives the result.

∎

We are now in a position to prove the Bianchi cube theorem for Darboux transforms in a self-dual symmetric space. This is motivated by Theorem 4.17 of [11] and involves little more than a combination of the permutability of dressing curved flats (Proposition 5.18) and the above corollary.

Corollary 5.22 (Bianchi cube)
Let f be isothermic in a self-dual symmetric R-space M. Let f_a, f_b, f_c be Darboux transforms $\mathcal{D}_a f, \mathcal{D}_b f, \mathcal{D}_c f$ respectively where a, b, c are distinct and generic complementarity relations are satisfied whenever required. We claim that there exist common Darboux transforms

$$f_{ab} = \mathcal{D}_a f_b = \mathcal{D}_b f_a, \quad f_{ac} = \mathcal{D}_a f_c = \mathcal{D}_c f_a, \quad f_{bc} = \mathcal{D}_b f_b = \mathcal{D}_c f_b,$$

such that there exists a simultaneous Darboux transform of all three (see figure 5-3):

$$\hat{f}_{abc} = \mathcal{D}_a f_{bc} = \mathcal{D}_b f_{ac} = \mathcal{D}_c f_{ab}.$$

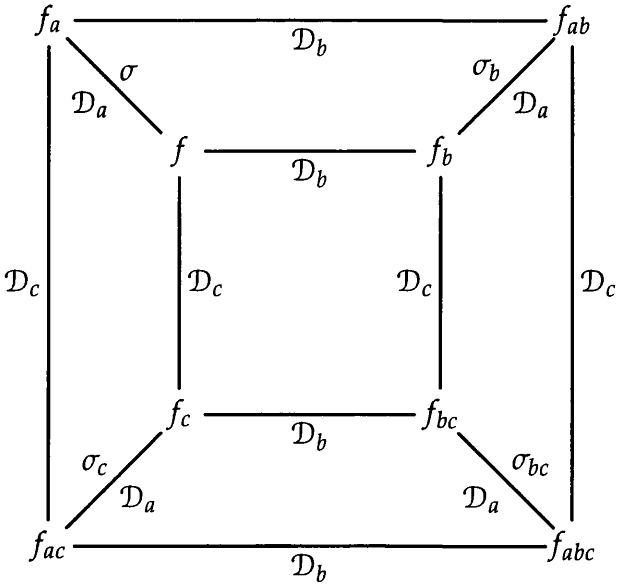

Figure 5-3: The Bianchi cube

Proof Suppose, without loss of generality, that $c/a, b/a < 1$ and let $\sigma = (f, f_a)$. By Theorem 5.20 there exist dressing transforms r_b, r_c, defined with respect to $\mathfrak{w}_{b,c} \subset \Sigma \times M$ such that $f_b = r_b f$, $f_c = r_c f$. Define new curved flats $\sigma_b = r_b\sigma$, $\sigma_c = r_c\sigma$. By Corollary 5.21 $\sigma_b = (f_b, f_{ab})$ where $f_{ab} = r_b f_a = \mathcal{D}_b f_a = \mathcal{D}_a f_b$ and similarly $\sigma_c = (f_c, f_{ac})$ where $f_{ac} = r_c f_a = \mathcal{D}_c f_a = \mathcal{D}_a f_c$. By Proposition 5.18 there exist dressing transforms r'_b, r'_c defined in terms of bundles of parabolic subalgebras $\mathfrak{w}'_{a,b}$ in M satisfying $r'_b r_c = r'_c r_b$ and such that $\sigma_{bc} := r'_b\sigma_c = r'_c\sigma_b$ is a curved flat. Since $\mathfrak{w}'_{b,c} \subset \Sigma \times M$ we know that r'_b, r'_c act by Darboux transforms and so $f_{bc} := r'_c f_{ab} = r'_b f_{ac}$ and $f_{abc} := r'_c f_b = r'_b f_c$ are simultaneous Darboux transforms as claimed. The definition of $\mathfrak{w}'_{b,c}$ as given in Proposition 5.18 assures us that the coefficients of the simultaneous Darboux transforms are exactly as claimed.

■

A statement of the Bianchi cube theorem for Bäcklund-type transforms of $\mathfrak{p}$-flat maps may be found in [11], whilst the corresponding statement for Ribaucour transforms of enveloping surfaces of a sphere congruence can be seen in [14].

Chapter 6

Isothermic Submanifolds: Examples and Further Results

6.1 Introduction

In this chapter we move away from the abstract discussion of Chapter 5 to consider in detail several examples of isothermic submanifolds of certain symmetric R-spaces, namely in the projective light-cones $\mathbb{P}(\mathcal{L}^n_{\mathbb{C}})$ (and their real forms), the Grassmannians $G_k(\mathbb{R}^n)$, and $\mathrm{SO}(n)$ as a homogeneous $\mathrm{SO}(n,n)$-space. Upon restricting to light-cones in a 6-dimensional vector space we are able to see that Definition 2.16 of an isothermic line congruence in $\mathbb{P}^3$ is exactly the condition on a map into the Klein quadric being isothermic in the sense of Chapter 5. We actually have more, for the Demoulin–Tzitzeica Theorem 2.15 is seen to hold in any light-cone (real or complex) in six dimensions: in particular Laplace transforms of an isothermic sphere congruence in S^3 are isothermic. Our investigation of isothermic submanifolds of the Grassmannians is mostly concerned with line congruences in $\mathbb{P}^{n-1}$ (2-submanifolds of $G_2(\mathbb{R}^n)$) where $n \geq 5$. While generic line congruences in $\mathbb{P}^{n-1}$ have no special structure in the sense of Chapter 2, we see that *isothermic* congruences possess almost all the structure of congruences in $\mathbb{P}^3$: focal surfaces, canonical co-ordinates, Laplace transforms, etc. The distinction between isothermic congruences in $\mathbb{P}^{n-1}$ and those in $\mathbb{P}^3$ is not great and we prove several theorems that get to the heart of the difference. Extending the discussion to 'maximal' isothermic submanifolds of the non-self-dual $G_k(\mathbb{R}^n)$, $k \neq n/2$, we observe a duality between isothermic congruences of k- and $(n-k)$-planes, which is seen to depend only on considerations of the kernel and image of the closed 1-form η (acting on $\mathbb{R}^n$). The duality motivates a substitute for the theorem of Bianchi permutability of Darboux transforms. As a second example of a non-self-dual symmetric R-space we investigate $\mathrm{SO}(n)$, n odd: exactly as in the Grassmannian discussion we observe a canonical duality between maximal isothermic submanifolds of $\mathrm{SO}(n)$ and

$SO(n)^*$, and a quasi-permutability theorem. The commonality of the two discussions suggests the existence of a general result for non-self-dual symmetric R-spaces which, as yet, remains hidden.

6.2 Light-cones and Line Congruences in $\mathbb{P}^3$ Revisited

Consider the complex light-cone in $\mathbb{C}^n$. Given a non-degenerate immersion $\ell : \Sigma \to \mathbb{P}(\mathcal{L}^n_{\mathbb{C}})$ (i.e. $\operatorname{Im} d\ell$ has exactly two null directions, but is non-isotropic), we pull back the null directions of the conformal structure to find two distinct line subbundles L_1, L_2 of $T^{\mathbb{C}}\Sigma = T\Sigma \otimes \mathbb{C}$. Let X, Y be commuting sections of these. In the absence of additional structure (either L_1, L_2 are complexifications of real bundles or $\overline{L_1} = L_2$) the existence of conformal co-ordinates cannot be supposed, but this is of no concern, for all discussion can be formulated in terms of X, Y. For brevity we will write $X\mu = \mu_X$ for any function μ with domain $\subset \Sigma$. For the duration of this section ω_1, ω_2 are the (closed) 1-forms dual to X, Y: i.e. $\omega_1 : T^{\mathbb{C}}\Sigma \to \mathbb{C}$ satisfies $\omega_1(X) = 1$, $\omega_1(Y) = 0$.

Let X, Y be commuting sections of the null directions of the conformal structure. Following [17] we let ℓ be the unique (up to scale) lift such that $(\ell_X, \ell_Y) = 1/2$. Furthermore let $\hat{\ell}$ be the unique section of the conformal Gauss map $S := \langle \ell, d\ell, \ell_{XY} \rangle$ such that $\hat{\ell} \perp \langle \ell_X, \ell_Y \rangle$ and $(\ell, \hat{\ell}) = -1$. Then

$$\operatorname{Proj}_S \xi = (\xi, \hat{\ell})\ell - 2(\xi, \ell_Y)\ell_X - 2(\xi, \ell_X)\ell_Y + (\xi, \ell)\hat{\ell}. \tag{6.1}$$

Define sections κ, κ' of the the weightless normal bundle $S^\perp$ by $\kappa = \operatorname{Proj}_{S^\perp} \ell_{XX}$, $\kappa' = \operatorname{Proj}_{S^\perp} \ell_{YY}$. From these and (6.1) we may calculate Willmore density of ℓ: defining S_X, S_Y similarly to Section 2.7 we have $\operatorname{tr}(S_X^* S_X) = 0 = \operatorname{tr}(S_Y^* S_Y)$ and

$$\operatorname{tr}(S_X^* S_Y) = \operatorname{coeff}_\kappa(\operatorname{Proj}_{S^\perp} \circ d_X \circ \operatorname{Proj}_S \kappa_Y) = 2(\kappa, \kappa'),$$

and so the Willmore density of ℓ is

$$\mathcal{E} = 2(\kappa, \kappa')\omega_1 \wedge \omega_2. \tag{6.2}$$

Now let ξ be a section of $S^\perp$ and $\nabla^\perp$ the connection on $S^\perp$ given by the restriction of flat d. By (6.1) we have

$$\begin{aligned}
&\nabla^\perp_X \xi = d_X \xi - \operatorname{Proj}_S d_X \xi = \xi_X + (\xi, \hat{\ell}_X)\ell - 2(\xi, \kappa)\ell_Y, \\
&\nabla^\perp_Y \xi = \xi_Y + (\xi, \hat{\ell}_Y)\ell - 2(\xi, \kappa')\ell_X. \\
&\therefore \nabla^\perp_X \nabla^\perp_Y \xi = \operatorname{Proj}_{S^\perp} d_X \nabla^\perp_Y \xi = \operatorname{Proj}_{S^\perp}(\xi_{XY} - 2(\xi, \kappa')\ell_{XX}),
\end{aligned}$$

and so

$$R^{\nabla^\perp}\xi = 2\left[(\xi,\kappa)\kappa' - (\xi,\kappa')\kappa\right]\omega_1 \wedge \omega_2, \tag{6.3}$$

exactly as in [17]. The condition on the curvature vanishing is then that κ and κ' are scalar multiples.

Restrict now to non-degenerate isothermic maps $\ell : \Sigma \to \mathbb{P}(\mathcal{L}_{\mathbb{C}})$. We have the following proposition.

Proposition 6.1
Given a non-degenerate isothermic surface $\Sigma \hookrightarrow \mathbb{P}(\mathcal{L}^n_{\mathbb{C}})$, there exist commuting sections X, Y of the null directions of the induced conformal structure on $T^{\mathbb{C}}\Sigma$ such that if $\ell : \Sigma \to \mathcal{L}^n_{\mathbb{C}}$ is the lift (up to sign) with $(\ell_X, \ell_Y) = 1/2$, then

$$\ell \wedge (\ell_Y\omega_1 + \ell_X\omega_2) \tag{6.4}$$

is closed, where ω_1, ω_2 are the (closed) 1-forms dual to X, Y. Moreover any *closed 1-form with values in $\ell \wedge \ell^\perp$ is a constant scalar multiple of (6.4).*

Remark: This resonates with Proposition 2.9 in [11]: given $\ell : \Sigma \to \mathbb{P}(\mathcal{L}^{n+1,1})$ isothermic, η is unique up to scale unless F is totally umbilic.

Proof The strategy of the proof will be to construct sections X, Y and a lift ℓ as described in the statement of the proposition starting from any choice of these and applying the isothermic condition $\mathrm{d}\eta = 0$.

Let $X, Y \in \Gamma T^*\Sigma$ be any commuting sections of the null directions of the conformal structure induced by ℓ and ω_1, ω_2 be the dual 1-forms to X, Y. The closed 1-form η takes values in $\ell \wedge \ell^\perp = \ell \wedge (\mathrm{d}\ell \oplus S^\perp)$. Write $\eta = A\omega_1 + B\omega_2$ for some $A, B \in \ell \wedge \ell^\perp$. The closedness condition reads $A_Y = B_X$. Let ℓ be the lift such that $(\ell_X, \ell_Y) = 1/2$ and write

$$A = \ell \wedge (\hat{p}\ell_X + p\ell_Y + \sigma), \quad B = \ell \wedge (q\ell_X + \hat{q}\ell_Y + \sigma'),$$

where $\sigma, \sigma' \in \Gamma S^\perp$ and $p, \ldots, \hat{q}$ are scalar functions. $A_Y = B_X$ now says that

$$\begin{aligned} &\hat{p}\ell_Y \wedge \ell_X + \ell_Y \wedge \sigma + \ell \wedge (\hat{p}_Y\ell_X + \hat{p}\ell_{XY} + p_Y\ell_Y + p\ell_{YY} + \sigma_Y) \\ &= \hat{q}\ell_X \wedge \ell_Y + \ell_X \wedge \hat{\sigma} + \ell \wedge (\hat{q}_X\ell_Y + \hat{q}\ell_{XY} + q_X\ell_X + q\ell_{XX} + \hat{\sigma}_X), \end{aligned}$$

which, by linear independence, yields

$$\hat{p} = \hat{q} = 0, \quad \sigma = \hat{\sigma} = 0, \quad p_Y = 0 = q_X, \quad q\kappa = p\kappa'.$$

Now $p \in \ker Y$, $q \in \ker X$, but p/q is a fixed function. Since $\ker X \cap \ker Y = \{\text{constants}\}$ it follows that the solutions p, q are defined up to scaling by the *same* constant.

Define new sections $\tilde{X} = p^{-1/2}X$, $\tilde{Y} = q^{-1/2}Y$ and the lift $\tilde{\ell} = (pq)^{-1/4}\ell$. It is a straightforward calculation to see that $\tilde{\ell} \wedge (\tilde{\ell}_{\tilde{Y}}\tilde{\omega}_1 + \tilde{\ell}_{\tilde{X}}\tilde{\omega}_2)$ is closed. Since p, q are defined only up to a constant scale it follows that any closed 1-form is a constant multiple of this.

■

From now on $X, Y, \ell, \omega_1, \omega_2$ will be precisely those lifts/sections appearing in Proposition 6.1. It follows that a non-degenerate isothermic surface in a light-cone has $\kappa' = \kappa$ so that the induced connection on the normal bundle is flat (6.3) and the Willmore density (6.2) is $2(\kappa, \kappa)\omega_1 \wedge \omega_2$.

In order work with real forms of $\mathcal{L}_{\mathbb{C}}$ we need to know that the concepts of isothermic in real and complex light-cones correspond.

Proposition 6.2
The complexification $\ell^{\mathbb{C}}$ of a map $\ell : \Sigma \to \mathbb{P}(\mathcal{L})$ into some real form $\mathcal{L}$ of $\mathcal{L}_{\mathbb{C}}$ is isothermic iff ℓ is.

Proof Let $\ell^{\mathbb{C}} : \Sigma \to \mathbb{P}(\mathcal{L}_{\mathbb{C}})$ be isothermic with closed 1-form $\eta_{\mathbb{C}}$ such that $\ell^{\mathbb{C}}$ is the complexification of a real map $\ell : \Sigma \to \mathbb{P}(\mathcal{L})$ into a real form of $\mathcal{L}_{\mathbb{C}}$. Then $\ell^{\mathbb{C}} \wedge \ell^{\perp}_{\mathbb{C}}$ gets a conjugation with respect to which $\eta_{\mathbb{C}}$ splits into real and imaginary parts:

$$\eta_{\mathbb{C}} = \eta_{\mathbb{R}} + i\eta_i,$$

both of which are closed. At least one of $\eta_{\mathbb{R}}, \eta_i$ is non-zero and so ℓ is isothermic. Indeed we can do slightly better if ℓ is non-degenerate, for by Proposition 6.1 we know that $\eta_{\mathbb{C}} = c\eta_{\mathbb{R}}$ (or $c\eta_i$ if $\eta_{\mathbb{R}} = 0$) for some complex constant c.

Conversely, if $\ell^{\mathbb{R}} : \Sigma \to \mathbb{P}(\mathcal{L}_{\mathbb{R}})$ is isothermic with closed 1-form η, then $\ell \wedge \ell^{\perp} \subset \ell^{\mathbb{C}} \wedge \ell^{\perp}_{\mathbb{C}}$ and so $(\ell^{\mathbb{C}}, \eta)$ is isothermic.

■

Now restrict to the non-degenerate quadric $\mathbb{P}(\mathcal{L}_{\mathbb{C}})$ in $\mathbb{P}(\mathbb{C}^6)$ which, by the complex Klein correspondence (cf. Section 2.2), is in bijective correspondence with the set of lines in $\mathbb{P}(\mathbb{C}^4)$. If $\ell : \Sigma^2 \to \mathbb{P}(\mathcal{L}_{\mathbb{C}^6})$ is a non-degenerate immersion (line congruence) then, since there exist commuting sections X, Y of the null directions of $(\mathrm{d}\ell, \mathrm{d}\ell)$, we can replicate the entire discussion of Chapter 2, replacing partial differentiation with respect to conformal co-ordinates x, y with the action of the sections X, Y. In particular there exist focal surfaces (2.3) f, g such that $\ell = f \wedge g$, each satisfying an equation of Laplace (2.8)

$$\mathrm{f}_{XY} + a\mathrm{f}_X + b\mathrm{f}_Y + c\mathrm{f} = 0, \tag{6.5}$$

and therefore the Laplace transforms (Definition 2.6) $\mathrm{f}^{-1} := \mathrm{f}_X + b\mathrm{f}$, $\mathrm{g} := \mathrm{f}_X + a\mathrm{f}$, $\mathrm{g}_1 = \mathrm{g}_X + a_1\mathrm{g}$, $\ell^1 = \mathrm{g} \wedge \mathrm{g}_1$, $\ell^{-1} = \mathrm{f} \wedge \mathrm{f}^{-1}$ are well-defined. One can check by hand that $\ell^{\pm 1}$ are indeed the null directions in the weightless normal bundle $S_{\mathbb{C}}^{\perp}$ (Theorem 2.13). Furthermore if ℓ is the complexification of a real line congruence, then $S_{\mathbb{C}} = S \otimes \mathbb{C}$ as one would expect. It follows that the Laplace transforms of ℓ are the complexifications of the Laplace transforms of the real form of ℓ. We also have that the structure equations of Section 2.9 hold, as does Definition 2.16 of isothermic line congruences. Let us be more explicit: let f be any lift of the focal surface f, define lifts $\mathrm{g}, \mathrm{f}^{-1}, \mathrm{g}^1$ in the usual way and the coefficient $\delta : \Sigma \to \mathbb{C}$ by $\mathrm{f}_{XX} \equiv \delta \mathrm{g}^1 \mod \mathrm{f}, \mathrm{g}, \mathrm{f}^{-1}$. The commuting sections X, Y of L_1, L_2 are *strictly isothermic conjugate* iff $\delta = -1$. The Demoulin–Tzitzeica Theorem 2.15 follows and we are in a position to prove the complex generalisation of Definition/Theorem 2.16.

Theorem 6.3 (cf. Theorem 2.16)
Suppose $\ell : \Sigma \to \mathbb{P}(\mathcal{L}_{\mathbb{C}}^6)$ is a non-degenerate immersion. Then ℓ is isothermic in the sense of Chapter 5 iff there exist commuting sections X, Y of the null directions of $(\mathrm{d}\ell, \mathrm{d}\ell)$ such that $(\ln \delta)_{XY} = 0$.

Proof Suppose $\ell : \Sigma \to \mathbb{P}(\mathcal{L}_{\mathbb{C}}^6)$ is isothermic. By Proposition 6.1 there exist commuting sections X, Y of the null directions of the conformal structure and a lift of ℓ such that $\eta = \ell \wedge (\ell_Y \omega_1 + \ell_X \omega_2)$ where ω_1, ω_2 are the closed 1-forms dual to X, Y. Let $\tilde{\mathrm{f}}$ be any lift of the focal surface f and $\tilde{\mathrm{g}} := \tilde{\mathrm{f}}_Y + a\tilde{\mathrm{f}}$. Let λ be a solution to $\lambda^{-4} = 2((\tilde{\mathrm{f}} \wedge \tilde{\mathrm{g}})_X, (\tilde{\mathrm{f}} \wedge \tilde{\mathrm{g}})_Y)$ and define a lift $\mathrm{f} = \lambda \tilde{\mathrm{f}}$. Defining g similarly gives a lift $\ell = \mathrm{f} \wedge \mathrm{g}$ which satisfies $(\ell_X, \ell_Y) = 1/2$. Define lifts $\mathrm{f}^{-1}, \mathrm{g}^1$ in terms of f in the usual way. By straightforward calculation we see that

$$\kappa = h\mathrm{f}^{-1} \wedge \mathrm{f} + \delta \mathrm{g}^1 \wedge \mathrm{g} = \kappa' = \gamma' \mathrm{f}^{-1} \wedge \mathrm{f} - \mathrm{g}^1 \wedge \mathrm{g},$$

where $h = ab - c + a_X$ is the coefficient of the Laplace invariant $\mathcal{H} = h\omega_1\omega_2$ of ℓ and $\mathrm{g}_Y^1 \equiv \gamma' \mathrm{f}^{-1} \mod \mathrm{f}, \mathrm{g}, \mathrm{g}^1$. Thus $\delta = -1$ and X, Y are strictly isothermic conjugate on f, g. ℓ is therefore isothermic in the sense of Definition 2.16.

Suppose now that ℓ is isothermic in the sense of Definition 2.16 so that $(\ln \delta)_{XY} = 0$ where X, Y are any commuting sections of the null directions of the conformal structure. Observe that a function $F : \Sigma \to \mathbb{C}$ is in the kernel of X iff $\mathrm{d}F = \hat{F}\omega_2$ where $\hat{F} \in \ker X$. We can therefore integrate $(\ln\delta)_X = \hat{p} \in \ker Y$, $(\ln\delta)_Y = \hat{q} \in \ker X$ to see that there exist functions $p \in \ker Y$, $q \in \ker X$ such that $\ln\delta = i\pi + 2\ln q - 2\ln p$. Similarly to the proof of the Demoulin–Tzitzeica Theorem 2.15 we see that with respect to the *commuting* sections qX, pY we have $\delta = -1$. We may therefore assume that X, Y are strictly isothermic on f, g. Choose lifts f, g again such that $(\ell_X, \ell_Y) = 1/2$: note that a rescaling of f leaves $\delta = -1$ unchanged. Recall the definition of the dual focal surfaces in section[1] 2.8 and observe, by (2.17), that

$$\eta := \mathsf{f} \otimes \mathsf{g}^*\,\omega_1 + \mathsf{g} \otimes \mathsf{f}^*\,\omega_2$$

is closed and takes values in $\mathrm{hom}(\mathbb{C}^4/\ell, \ell)$. ℓ is therefore isothermic in the symmetric R-space $G_2(\mathbb{C}^4) = \mathrm{SL}(4, \mathbb{C})/P$.

■

Observe that in view of the isomorphism $\mathfrak{sl}(4, \mathbb{C}) \cong \mathfrak{so}(6, \mathbb{C})$ given by

$$A \in \mathfrak{sl}(4, \mathbb{C}) : \mathsf{u} \wedge \mathsf{v} \mapsto (A\mathsf{u}) \wedge \mathsf{v} + \mathsf{u} \wedge (A\mathsf{v})$$

(the derivative of (2.1)), we see that, viewed as a 1-form with values in $\mathfrak{so}(6, \mathbb{C})$, we have

$$\eta = 2\ell \wedge (\ell_Y\,\omega_1 + \ell_X\,\omega_2),$$

as expected.

We could have obtained a proof of Theorem 2.16 much more straightforwardly by working from the start in the symmetric R-space $\mathrm{SL}(4, \mathbb{R})/P$. Similarly to the proof of Proposition 6.1, it is not difficult to see that if $\ell = \mathsf{f} \wedge \mathsf{g}$ is isothermic in the sense of Chapter 5, where f is any lift of the focal surface f, then there exist functions λ, μ on Σ such that

$$\eta = \lambda \mathsf{f} \otimes \mathsf{g}^*\,\mathrm{d}x + \mu \mathsf{g} \otimes \mathsf{f}^*\,\mathrm{d}y, \quad \text{where}$$
$$\lambda_y = 0 = \mu_x, \qquad \lambda = -\delta\mu, \qquad \lambda\gamma' = -h\mu.$$

The final three equations tell us that $W = 0$ and that $(\ln\delta)_{xy} = 0$. ℓ is therefore isothermic in the sense of Definition 2.16. Conversely the above equations for λ, μ are integrable iff $W = 0 = (\ln\delta)_{xy}$, thus η so defined is closed iff ℓ is isothermic in the sense of Definition 2.16. Working in the generalised complex setting tells us more, as

[1] E.g. $\mathsf{f}^* \in \Gamma\,\mathrm{Ann}(f, g, f^{-1})$ such that $\mathsf{f}^*\mathfrak{g}_1 = 1$.

the following corollary shows.

Corollary 6.4
Laplace transforms of (complex) isothermic line congruences are isothermic.

Proof We know that $\eta = \mathrm{f} \otimes \mathrm{g}^* \,\omega_1 + \mathrm{g} \otimes \mathrm{f}^* \,\omega_2$ in terms of some lift of f and closed duals ω_1, ω_2 to strictly isothermic X, Y, the dual surfaces $\mathrm{f}^* \in \Gamma\,\mathrm{Ann}(f, df)$, $\mathrm{g}^* \in \Gamma\,\mathrm{Ann}(g, dg)$ being defined so that $\mathrm{f}^*\mathrm{g}^1 = 1 = \mathrm{g}^*\mathrm{f}^{-1}$. In Section 2.8 we calculated the conjugate net equation of f^* and its Laplace transform:

$$\mathrm{f}^*_{XY} - a\mathrm{f}^*_X - b\mathrm{f}^*_Y - (h - ab + b_Y)\mathrm{f}^* = 0, \qquad \mathrm{f}^{-1}_* = \mathrm{f}^*_Y - a\mathrm{f}^*.$$

The four focal surfaces $\mathrm{f}, \mathrm{f}^{-1}, \mathrm{f}^*, \mathrm{f}^{-1}_*$ therefore satisfy

$$\begin{aligned} \mathrm{f}_X &= \mathrm{f}^{-1} - b\mathrm{f}, & \mathrm{f}^{-1}_Y &= -a\mathrm{f}^{-1} + h^{-1}\mathrm{f}, \\ \mathrm{f}^*_Y &= \mathrm{f}^*_{-1} + a\mathrm{f}^*, & (\mathrm{f}^{-1}_*)_X &= b\mathrm{f}^{-1}_* + h^{-1}\mathrm{f}^*, \end{aligned}$$

from which we see that

$$\eta_{-1} := \mathrm{f}^{-1} \otimes \mathrm{f}^* \,\omega_1 + \mathrm{f} \otimes \mathrm{f}^{-1}_* \,\omega_2$$

is closed and takes values in $\hom(\mathbb{C}^6/\ell^{-1}, \ell^{-1})$.

∎

It is easy to see that the lifts of the dual surfaces $\mathrm{f}^* \in \Gamma\,\mathrm{Ann}(\mathrm{g}, \mathrm{f}, \mathrm{f}^{-1})$, $\mathrm{f}^{-1}_* \in \Gamma\,\mathrm{Ann}(\mathrm{f}, \mathrm{f}^{-1}, \mathrm{f}^{-2})$ are normalised by $\mathrm{f}^*\mathrm{f}^{-2} = -1 = \mathrm{f}^{-1}_*\mathrm{g}$ (via the 7^{th} structure equation; Section 2.9).

In particular the corollary tells us something we did not already know: in Lie sphere geometry ($\mathbb{P}(\mathcal{L}^{4,2})$) the Laplace transforms of an isothermic sphere congruence in S^3 are isothermic. When the induced conformal structure is definite these are real sphere congruences. Similarly one has isothermic Laplace transforms of isothermic surfaces in $\mathbb{P}(\mathcal{L}^{5,1})$ although these are always complex conjugate.

6.3 Line Congruences in Higher Dimensions

Instead of discussing Darboux transforms of isothermic line congruences in $\mathbb{P}^3$, we now consider congruences in $\mathbb{P}^n = \mathbb{P}(\mathbb{R}^n)$ in terms of Darboux pairs. *A priori* a 2-parameter family of lines ℓ in $\mathbb{P}^n$ ($n \geq 4$) has no focal surfaces: the condition (2.4) on the existence of focal surfaces is not a statement about top-degree forms and so is no longer a single quadratic condition. We cannot therefore define a congruence in terms of focal surfaces as we did in $\mathbb{P}^3$. With isothermic congruences however progress can be made, indeed we shall see that much of the structure (focal surfaces, special co-ordinates, Laplace transforms) one sees in $\mathbb{P}^3$ is present. The following analysis can be very easily extended to line congruences in $\mathbb{P}(\mathbb{C}^n)$ in a similar fashion to the previous section. Much of the discussion indeed extends to isothermic submanifolds of any Grassmannian $G_k(\mathbb{R}^n)$ and to complex Grassmannians. We restrict for the moment to real line congruences in order to more easily see how much of the structure described in Chapter 2 is preserved.

Isothermic congruences have Darboux transforms: let $\hat{\ell} : \Sigma \to G_{n-2}(\mathbb{R}^n)$ be a Darboux transform of (ℓ, η), then $\sigma = (\ell, \hat{\ell}) : \Sigma \to G/K = \mathrm{SL}(n)/\mathrm{S}(\mathrm{GL}(2) \times \mathrm{GL}(n-2))$ is a curved flat and so each $\operatorname{Im} \mathrm{d}\sigma$ is a maximal Abelian subalgebra

$$\mathfrak{a} \subset \hom(\ell, \hat{\ell}) \oplus \hom(\hat{\ell}, \ell) \subset \mathfrak{sl}(n)$$

transverse to both parts. Make the usual assumption that each $\mathfrak{a}$ is semisimple, and all are in a single conjugacy class: from now on we will refer to a line congruence as *maximal* if there exists a Darboux transform such that this holds. Similarly η will be referred to as *maximally diagonal*. We see that $\{T^2 : T \in \mathfrak{a}\}$ are simultaneously diagonalisable over ℓ and $\hat{\ell}$. There therefore exist (possibly complex conjugate) line subbundles[2] $L_i \subset \ell, \hat{L}_i \subset \hat{\ell}$, $i = 1, 2$ and isomorphisms $\chi_i : L_i \to \hat{L}_i$ such that

$$\mathfrak{a} = \left\langle \chi_1 + \chi_1^{-1} \right\rangle \oplus \left\langle \chi_2 + \chi_2^{-1} \right\rangle.$$

For concreteness we will restrict to the case where the L_i are real; one can easily repeat the analysis for complex line congruences similarly to the previous section. To detect the line bundles $L_i, \hat{L}_i$ from $\mathfrak{a}$ observe that everything in $\mathfrak{a}$ has rank 2 except each $\chi_i + \chi_i^{-1}$. Since $\mathrm{d}\sigma$ is the Maurer–Cartan form $\mathcal{N}$ with respect to the isomorphism $TG/K \cong \mathfrak{a}$ (Chapter 1) it follows that there exist 1-forms ω_i such that

$$\mathcal{N} = \sum_{i=1}^{2} \omega_i (\chi_i + \chi_i^{-1}).$$

$\mathcal{N}$ satisfies the Codazzi equation $\mathrm{d}^{\mathcal{D}} \mathcal{N} = 0$ and so we have

$$0 = \sum_{i=1}^{2} \mathrm{d}\omega_i (\chi_i + \chi_i^{-1}) + \omega_i \wedge \mathcal{D}(\chi_i + \chi_i^{-1}).$$

Take traces by evaluating the above on the line bundles $L_i, \hat{L}_i$ to see that $\mathcal{D}(\chi_i + \chi_i^{-1}) \perp$

[2] Not to be confused with the line subbundles L_1, L_2 of $T^{\mathbb{C}}\Sigma$ described in our discussion of light-cones. Since only $G_2(\mathbb{R}^4)$ is a light-cone the notation only pairs up in one situation.

$(\chi_j + \chi_j^{-1})$ with respect to the Killing form $\forall i, j$, and that $(\chi_1 + \chi_1^{-1}) \perp (\chi_2 + \chi_2^{-1})$. Since the $\left\langle \chi_i + \chi_i^{-1} \right\rangle$ are non-isotropic, the Codazzi equation decouples to give

$$\mathrm{d}\omega_1 = 0 = \mathrm{d}\omega_2 = \omega_1 \wedge \mathcal{D}(\chi_1 + \chi_1^{-1}) + \omega_2 \wedge \mathcal{D}(\chi_2 + \chi_2^{-1}).$$

The ω_i can be locally integrated to give local co-ordinates x, y ($\mathrm{d}x = \omega_1$, $\mathrm{d}y = \omega_2$) and so we can choose the 1-form η to be

$$\eta = \mathrm{d}\hat{\ell} = \chi_1^{-1}\mathrm{d}x + \chi_2^{-1}\mathrm{d}y, \qquad \mathrm{d}\ell = \chi_1\mathrm{d}x + \chi_2\mathrm{d}y.$$

The third Codazzi equation simply expresses the closedness of both $\eta, \mathrm{d}\ell$ and so tells us nothing new. It is implicit in the choice of scale of η that $\hat{\ell} = \mathcal{D}_{-1}\ell$.

The particular choice of η and of the co-ordinates x, y (defined up to an additive constant) are dependent on the Darboux transform $\hat{\ell}$. However, the line bundles L_i are independent of $\hat{\ell}$ and are detected from η as the images of the only rank 1 elements in $\eta(T\Sigma)$. In possession of these bundles we can invoke Frobenius' theorem or the existence of isothermal co-ordinates (Section 2.5) to find co-ordinates x, y defined up to scale. Note that η as defined above takes values in $\hom(\hat{\ell}, \ell)$ rather than $\hom(\mathbb{R}^n/\ell, \ell)$. However, in view of the discussion of Section 5.4, there exists a unique linear isomorphism $\hat{\ell} \cong \mathbb{R}^n/\ell$ with respect to which η is quotient-valued. Once $\hat{\ell}$ is fixed, the χ_i^{-1} specify, as their common kernel, a complement $\hat{L}_0$ to $\hat{L}_1 \oplus \hat{L}_2 \subset \hat{\ell}$. To summarise,

$$\Sigma \times \mathbb{R}^n = \underbrace{L_1 \oplus L_2}_{\ell} \oplus \underbrace{\hat{L}_0 \oplus \hat{L}_1 \oplus \hat{L}_2}_{\hat{\ell}}.$$

More is true, for we have actually stumbled on a duality between isothermic congruences of 2- and of $(n-2)$-planes (this will be generalised in the next section). The bundle $\tilde{\ell} := \ell \oplus \hat{L}_0 \in \Sigma \times G_{n-2}(\mathbb{R}^n)$ is independent of the choice of $\hat{\ell}$ since $\tilde{\ell} = \bigcap_{X \in T\Sigma} \ker \eta_X$. Consequently $\ell \oplus \hat{L}_0$ is an isothermic submanifold of $G_{n-2}(\mathbb{R}^n)$ with closed 1-form η viewed as taking values in $\hom(\mathbb{R}^n/(\ell \oplus \hat{L}_0), \ell \oplus \hat{L}_0)$. We can pass from $\tilde{\ell}$ back to ℓ by observing that $\ell = \bigcup_{X \in T\Sigma} \eta_X(\mathbb{R}^n)$. Note that the duality $\ell \mapsto \tilde{\ell}$ is the identity when we are in the self-dual situation of isothermic congruences of 2-planes in $\mathbb{R}^4$. More is true, for we have the following theorem.

Theorem 6.5

Maximal isothermic line congruences in $\mathbb{P}^n$ have focal surfaces on which the lines describe conjugate nets. Isothermic congruences then have Laplace invariants, and well-defined Laplace transforms.

Proof Let $\mathrm{f} \in \Gamma L_1$ and notice that $\mathrm{f}_y = \mathrm{d}\ell(\partial_y)\mathrm{f} = \chi_2 \mathrm{f} \equiv 0 \mod \ell$. Thus L_1 (similarly L_2) is a focal surface for ℓ with $L_2 \subset \langle L_1, (L_1)_y \rangle$ and $L_1 \subset \langle L_2, (L_2)_x \rangle$. Any lift $\mathrm{f} \in \Gamma L_1$ therefore satisfies a conjugate net Laplace equation (cf. (2.8), (6.5)):

$$\mathrm{f}_{xy} + a\mathrm{f}_x + b\mathrm{f}_y + c\mathrm{f} = 0.$$

Making the usual definition of a lift g of L_2: $\mathrm{g} := \mathrm{f}_y + a\mathrm{f}$ we see that g satisfies the conjugate net equation

$$\mathrm{g}_{xy} + a^1\mathrm{g}_x + b\mathrm{g}_y + c^1\mathrm{g} = 0,$$

where a^1, c^1 are defined as in Section 2.3. The *Laplace invariant* $\mathcal{H} := (ab - c + a_x)\,\mathrm{d}x\mathrm{d}y$ of ℓ is well defined and invariant of the choice of lift f. Furthermore, the conjugate net equations above tell us that

$$\mathrm{f}^{-1} := \mathrm{f}_x + b\mathrm{f}, \qquad \mathrm{g}^1 := \mathrm{g}_y + a^1\mathrm{g}$$

are the focal surfaces of new line congruences $\ell^{-1} = \langle \mathrm{f}, \mathrm{f}^{-1} \rangle$, $\ell^1 = \langle \mathrm{g}, \mathrm{g}^1 \rangle$.

∎

We are almost in a position to be able to prove a Demoulin–Tzitzeica-type theorem (cf. Theorem 2.15).

Theorem 6.6
The Laplace transforms of a maximal isothermic congruence are isothermic and so (generically) possess Laplace transforms of their own. The entire Laplace sequence of an isothermic congruence is therefore isothermic.

To prove this theorem, we will describe η, similarly to the previous section, in terms of natural *dual surfaces* to L_1, L_2. Let $\hat{\ell}$ be a fixed Darboux transform of ℓ. Choose a lift $\mathrm{f} \in \Gamma L_1$ and define $\mathrm{g} = \mathrm{f}_y + a\mathrm{f}$, $\hat{\mathrm{f}} = \chi_1 \mathrm{f} \in \Gamma \hat{L}_1$, $\hat{\mathrm{g}} := \chi_2 \mathrm{g} \in \Gamma \hat{L}_2$. Observe that

$$\mathrm{f}_x = \mathrm{d}\ell(\partial_x)\mathrm{f} = \mathcal{D}_{\partial_x}\mathrm{f} + \mathcal{N}_{\partial_x}\mathrm{f} \equiv \chi_1 \mathrm{f} \equiv \hat{\mathrm{f}} \mod \ell \Rightarrow \mathrm{f}^{-1} \equiv \hat{\mathrm{f}} \mod \ell. \tag{6.6}$$

Similarly $\mathrm{g}^1 \equiv \hat{\mathrm{g}} \mod \ell$. Since $\hat{L}_0$ is independent (mod ℓ) of the choice of Darboux transform, it follows that $\mathbb{R}^n = \langle \mathrm{f}^{-1}, \mathrm{g}^1 \rangle \oplus (\ell \oplus \hat{L}_0)$ is a decomposition *independent* of the Darboux transform $\hat{\ell}$. We can therefore define coefficients $*, *', \delta, \gamma'$, similarly to (2.10,2.12) and independent of $\hat{\ell}$, such that

$$\mathrm{f}^{-1}_x \equiv *\mathrm{f}^{-1} + \delta \mathrm{g}^1, \quad \mathrm{g}^1_y = \gamma' \mathrm{f}^{-1} + *'\mathrm{g}^1 \mod \ell \oplus \hat{L}_0. \tag{6.7}$$

Since

$$\begin{aligned}\chi_2^{-1}\mathrm{f}_x^{-1} &= -(\chi_2^{-1})_x\mathrm{f}^{-1} = -(\chi_1^{-1})_y\mathrm{f}^{-1} = -(\chi_1^{-1}\mathrm{f}^{-1})_y + \chi_1^{-1}\mathrm{f}_y^{-1} \\ &= -\mathrm{f}_y - a\mathrm{f} = -\mathrm{g}\end{aligned} \tag{6.8}$$

we have $\delta = -1$ (similarly $-\gamma' = h := ab - c + a_x$). The linear maps

$$\mathrm{f}^* : \begin{pmatrix} \mathrm{f} \\ \mathrm{g} \\ \mathrm{f}^{-1} \\ \mathrm{g}^1 \\ \hat{L}_0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathrm{g}^* : \begin{pmatrix} \mathrm{f} \\ \mathrm{g} \\ \mathrm{f}^{-1} \\ \mathrm{g}^1 \\ \hat{L}_0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{6.9}$$

are well-defined, independent of $\hat{\ell}$, and satisfy

$$\chi_1^{-1} = \mathrm{f} \otimes \mathrm{g}^*, \qquad \chi_2^{-1} = \mathrm{g} \otimes \mathrm{f}^*.$$

The *dual surfaces* f^*, g^* are the annihilators of $\langle \mathrm{f}, \mathrm{df}, \hat{L}_0 \rangle$, $\langle \mathrm{g}, \mathrm{dg}, \hat{L}_0 \rangle$ respectively. The closedness condition $(\chi_1^{-1})_y = (\chi_2^{-1})_x$ tells us that the dual surfaces satisfy

$$\mathrm{f}_x^* = b\mathrm{f}^* + \mathrm{g}^*, \qquad \mathrm{g}_y^* = h\mathrm{f}^* + a\mathrm{g}^*. \tag{6.10}$$

The *dual congruence* $\ell^* := \langle \mathrm{f}^*, \mathrm{g}^* \rangle : \Sigma \to G_2(\mathbb{R}^n_*)$ is the annihilator of $\ell \oplus \hat{L}_0$. By (6.10) $\mathrm{f}^*, \mathrm{g}^*$ are focal for ℓ^*, and furthermore ℓ^* is an isothermic submanifold of $G_2(\mathbb{R}^n_*)$ with respect to the *same* η viewed as a 1-form with values in $\hom(\mathbb{R}^n_*/\ell^*, \ell^*)$:

$$\eta = \mathrm{g}^* \otimes \mathrm{f}\, dx + \mathrm{f}^* \otimes \mathrm{g}\, dy \in \Omega^1_\Sigma \otimes \hom(\mathbb{R}^n_*/\ell^*, \ell^*).$$

(6.10) also implies that $\mathrm{f}^*, \mathrm{g}^*$ satisfy conjugate net equations and so we may define Laplace transforms

$$\mathrm{f}_{-1}^* = \mathrm{f}_y^* - a\mathrm{f}^* \quad \text{since} \quad \mathrm{f}_{xy}^* - a\mathrm{f}_x^* - b\mathrm{f}_y^* - (h - ab + b_y)\mathrm{f}^* = 0.$$

Proof of Theorem 6.6 The dual surfaces $\mathrm{f}^*, \mathrm{f}_{-1}^*$ are easily seen to vanish on ℓ^{-1}. Furthermore, from the definition of f^{-1} and f_{-1}^* we see that $\mathrm{f}, \mathrm{f}^{-1}$ and their dual surfaces satisfy exactly the relations in Corollary 6.4 and so

$$\eta^{-1} := -\mathrm{f}^{-1} \otimes \mathrm{f}^* dx - \mathrm{f} \otimes \mathrm{f}_{-1}^* dy$$

is a closed 1-form with values in $\hom(\mathbb{R}^n/\ell^{-1}, \ell^{-1})$. Laplace transforms of isothermic line congruences are therefore isothermic.[3] Moreover the construction of η_{-1} shows that ℓ^{-1} is maximal isothermic, supposing that f^{-1} is not a degenerate surface, and so ℓ^{-2} and the entire Laplace sequence is also isothermic.

■

We have shown that a maximal isothermic line congruence in $\mathbb{P}^n$ comes equipped with almost all the trappings of a maximal line congruence in $\mathbb{P}^3$: co-ordinates, Laplace invariants and transforms, etc. While we cannot construct the conformal invariants $\mathcal{E}, \mathcal{R}$ of Section 2.7 for $n \geq 4$, one classical construction is so far absent: the Weingarten invariant. For this we require second fundamental forms on the focal surfaces L_1, L_2. In general a surface in $\mathbb{P}^n$ has an $(n-2)$-parameter family of conformal classes of second fundamental forms, the dimension of $\operatorname{Ann}(f, \mathrm{d}f)$ being $n-2$. There is however no expectation that this family forms a single conformal class. Indeed the following argument shows that generically this fails.

Proposition 6.7
Suppose that the collection of second fundamental forms on a focal surface L_1 of a maximal isothermic line congruence ℓ forms a single conformal class. Then the span $\langle \mathsf{f}, \mathsf{g}, \mathsf{f}^{-1}, \mathsf{g}^1 \rangle$ is a constant (d-parallel) 4-plane and so the line congruence lives in a $\mathbb{P}^3$.

Proof Fix a Darboux transform $\hat{\ell} = \hat{L}_0 \oplus \hat{L}_1 \oplus \hat{L}_2$, choose a lift $\mathsf{f} \in \Gamma L_1$, let $\mathsf{g} = \mathsf{f}_y + a\mathsf{f}$ be defined as above and set $\hat{\mathsf{f}} = \chi_1 \mathsf{f}$, $\hat{\mathsf{g}} = \chi_2 \mathsf{g}$. Choose sections $\hat{\mathsf{h}}_i \in \Gamma \hat{L}_0$. We have already seen (6.6) that

$$L_1^{(1)} := L_1 \oplus \operatorname{Im} \mathrm{d} L_1 = L_1 \oplus L_2 \oplus \hat{L}_1.$$

Similarly

$$L_2^{(1)} = L_1 \oplus L_2 \oplus \hat{L}_2, \quad \hat{L}_1^{(1)} \subset L_1 \oplus \hat{L}_0 \oplus \hat{L}_1 \oplus \hat{L}_2, \quad \hat{L}_2^{(1)} \subset L_2 \oplus \hat{L}_0 \oplus \hat{L}_1 \oplus \hat{L}_2.$$

Since $\hat{\mathsf{f}}_x \equiv \chi_1^{-1} \hat{\mathsf{f}} = \mathsf{f} \mod \hat{\ell}$, etc. we may define coefficients $\alpha, \ldots, \varepsilon_i'$ $(i = 1, \ldots, n-4)$ such that

$$\begin{aligned} \mathsf{f}_x &= \alpha \mathsf{f} + \beta \mathsf{g} + \hat{\mathsf{f}}, & \hat{\mathsf{f}}_x &= \mathsf{f} + \gamma \hat{\mathsf{f}} + \delta \hat{\mathsf{g}} + \varepsilon_i \hat{\mathsf{h}}_i, \\ \mathsf{g}_y &= \alpha' \mathsf{f} + \beta' \mathsf{g} + \hat{\mathsf{g}}, & \hat{\mathsf{g}}_y &= \mathsf{g} + \gamma' \hat{\mathsf{f}} + \delta' \hat{\mathsf{g}} + \varepsilon_i' \hat{\mathsf{h}}_i. \end{aligned} \tag{6.11}$$

It is not hard to check that δ, γ' are exactly as defined in (6.7). We calculate

$$\mathsf{f}_{xx} \equiv \delta \hat{\mathsf{g}} + \varepsilon_i \hat{\mathsf{h}}_i \mod L_1^{(1)}, \qquad \mathsf{f}_{yy} \equiv \mathsf{g}_y \equiv \hat{\mathsf{g}} \mod L_1^{(1)}.$$

[3] The -ve sign is for consistency, since $\mathsf{f}^* \mathsf{f}^{-2} = -1 = \mathsf{f}_{-1}^* \mathsf{g}$.

If all the second fundamental forms given by a choice f^* of dual line in $\mathbb{P}^3$ which annihilates $L_1^{(1)}$ coincide, we must have all $\varepsilon_i = 0$. By differentiating $\hat{\mathrm{f}} = \chi_1 \mathrm{f}$, etc. we see that

$$\hat{\mathrm{f}}_y = -a\hat{\mathrm{f}} - \beta\hat{\mathrm{g}}, \qquad \hat{\mathrm{g}}_x = -\alpha'\hat{\mathrm{f}} - b\hat{\mathrm{g}}. \tag{6.12}$$

Differentiating our above expression (6.11) for $\hat{\mathrm{f}}_x$ with respect to y and applying $\hat{\mathrm{f}}_y = -a\hat{\mathrm{f}} - \beta\hat{\mathrm{g}}$ it is a calculation to see that $\varepsilon_i = 0,\ \forall i \Rightarrow \varepsilon_i' = 0,\ \forall i$. We therefore have

$$\mathrm{d}\hat{\mathrm{f}} \subset \langle \mathrm{f}, \hat{\mathrm{f}}, \hat{\mathrm{g}} \rangle, \qquad \mathrm{d}\hat{\mathrm{g}} \subset \langle \mathrm{g}, \hat{\mathrm{f}}, \hat{\mathrm{g}} \rangle,$$

and so $\langle \mathrm{f}, \mathrm{g}, \mathrm{f}^{-1}, \mathrm{g}^1 \rangle = \langle \mathrm{f}, \mathrm{g}, \hat{\mathrm{f}}, \hat{\mathrm{g}} \rangle$ is constant. It follows that all Laplace transforms of ℓ are lines in $\mathbb{P}^3 := \mathbb{P}\langle \mathrm{f}, \mathrm{g}, \mathrm{f}^{-1}, \mathrm{g}^1 \rangle$.

■

When ℓ is maximal we can however choose specific second fundamental forms on the focal surfaces by letting $f^* := \mathrm{Ann}(\hat{L}_0 \oplus L_1^{(1)})$, g^* be the dual surfaces to L_1, L_2 as defined in (6.9). It is easy to see that

$$\mathrm{I\!I}_1 := f^*(\mathrm{f}_{xx}\mathrm{d}x^2 + 2\mathrm{f}_{xy}\mathrm{d}x\mathrm{d}y + \mathrm{f}_{yy}\mathrm{d}y^2) = \langle \delta\mathrm{d}x^2 + \mathrm{d}y^2 \rangle, \qquad \mathrm{I\!I}_2 = \langle h\mathrm{d}x^2 + \gamma'\mathrm{d}y^2 \rangle$$

and we can form the Weingarten invariant $\mathcal{W} = W\mathrm{d}x\mathrm{d}y = (h - \delta\gamma')\mathrm{d}x\mathrm{d}y$. However we have already seen from (6.8) that $\delta = -1, \gamma' = -h$ and so ℓ is a W-congruence: the asymptotic directions with respect to $\mathrm{I\!I}_1, \mathrm{I\!I}_2$ on the focal surfaces L_1, L_2 coincide.

We now consider a fixed Darboux transform $\hat{\ell} = \hat{L}_0 \oplus \hat{L}_1 \oplus \hat{L}_2$ and investigate its properties. Observe first that $\hat{\eta} = \chi_1\mathrm{d}x + \chi_2\mathrm{d}y$ can be viewed as taking values in $\hom(\mathbb{R}^n/(\hat{L}_1 \oplus \hat{L}_2), \hat{L}_1 \oplus \hat{L}_2)$ and so $(\hat{L}_1 \oplus \hat{L}_2, \hat{\eta})$ is an isothermic line congruence. By differentiating (6.12) we get conjugate net equations for $\hat{L}_1, \hat{L}_2$ and so can define Laplace transforms, isothermic by Theorem 6.6. We also get a Laplace invariant $\hat{\mathcal{H}} = \alpha'\beta\,\mathrm{d}x\mathrm{d}y$. We can define dual surfaces $\hat{f}^* = \mathrm{Ann}(\hat{L}_0, \hat{L}_1^{(1)})$, $\hat{g}^* = \mathrm{Ann}(\hat{L}_0, \hat{L}_2^{(1)})$ and specific second fundamental forms

$$\hat{\mathrm{I\!I}}_1 := \hat{f}^*(\hat{\mathrm{f}}_{xx}\mathrm{d}x^2 + 2\hat{\mathrm{f}}_{xy}\mathrm{d}x\mathrm{d}y + \hat{\mathrm{f}}_{yy}\mathrm{d}y^2), \qquad \hat{\mathrm{I\!I}}_2 := \hat{g}^*(\hat{\mathrm{g}}_{xx}\mathrm{d}x^2 + 2\hat{\mathrm{g}}_{xy}\mathrm{d}x\mathrm{d}y + \hat{\mathrm{g}}_{yy}\mathrm{d}y^2).$$

Since the closedness of η reads

$$\mathrm{f}_y\mathrm{g}^* + \mathrm{f}\mathrm{g}^*_y = \mathrm{g}_x\mathrm{f}^* + \mathrm{g}\mathrm{f}^*_x,$$

and $\operatorname{Im} d\hat{L}_0 = \operatorname{Im} \mathcal{D}\hat{L}_0 \subset \operatorname{Im} \mathcal{D}\hat{\ell} \subset \hat{\ell}$, it follows that

$$(\hat{L}_0)_x \subset \ker f^* \cap \hat{\ell} = \hat{L}_0 \oplus \hat{L}_1, \qquad (\hat{L}_0)_y \subset \ker g^* \cap \hat{\ell} = \hat{L}_0 \oplus \hat{L}_2.$$

Using this, and (6.11), we easily calculate that the second fundamental forms are

$$\hat{\mathrm{I\!I}}_1 = \langle dx^2 - dy^2 \rangle = \hat{\mathrm{I\!I}}_2,$$

thus confirming that $\hat{L}_1 \oplus \hat{L}_2$ is also a W-congruence. Indeed choosing lifts $\hat{f}^*, \hat{g}^*$ of the dual surfaces normalised such that $\hat{f}^* g = 1 = \hat{g}^* f$ yields

$$\hat{\eta} := \hat{f} \otimes \hat{g}^* dx + \hat{g} \otimes \hat{f}^* dy \, (= d\ell).$$

Finally observe that $\operatorname{Ann}(\ell)$ is isothermic with respect to η and that $(\operatorname{Ann}(\ell), \operatorname{Ann}(\hat{\ell}))$ are a Darboux pair, for they are certainly complementary, and, since $\hat{\ell} = \mathfrak{D}_{-1}\ell$, we have

$$((d - \eta)_X \operatorname{Ann}(\hat{\ell}))\hat{\ell} = d_X^{-1}(\operatorname{Ann}(\hat{\ell})\hat{\ell}) - \operatorname{Ann}(\hat{\ell})(d_X^{-1}\hat{\ell}) = 0,$$

thus $d_X^{-1} \operatorname{Ann}(\hat{\ell}) \subset \operatorname{Ann}(\hat{\ell})$. The correspondence is not as tight as one might expect however, for $\hat{L}_0 \oplus \ell$ is *not* a Darboux transform of $\hat{L}_1 \oplus \hat{L}_2$.

Proposition 6.8
Suppose $\hat{L}_0 \oplus \ell$ is a Darboux transform of $\hat{L}_1 \oplus \hat{L}_2$. Then $\hat{L}_0$ is constant and we are working in a $\mathbb{P}^3$.

Proof For this we would require that sections of $\hat{L}_0$ differentiate into $\langle \hat{L}_0, L_1, L_2 \rangle$ which is equivalent to $\hat{L}_0$ being constant. $\ell, \hat{L}_1 \oplus \hat{L}_2$ can be viewed as line congruences in $\mathbb{P}^3 := \mathbb{P}(\mathbb{R}^n / \hat{L}_0)$ and, since $\hat{L}_0$ is constant, all Laplace transforms of $\ell, \hat{L}_1 \oplus \hat{L}_2$ also live in $\mathbb{P}^3$.

∎

We may also ask questions of the cross-congruences $L_i \oplus \hat{L}_i$. Since $\hat{L}_i \subset L_i^{(1)}$ we see that L_i is focal for $L_i \oplus \hat{L}_i$. Similarly to our previous discussions, the converse holds iff the story belongs to a $\mathbb{P}^3$.

Proposition 6.9
$\hat{L}_i$ is focal for the cross-congruence $L_i \oplus \hat{L}_i$ iff $\ell \oplus \hat{L}_1 \oplus \hat{L}_2$ is constant.

Proof From our expressions (6.11,6.12) for the derivatives of $\hat{f}$ we see that there exist coefficients p, q such that $f \equiv p\hat{f}_x + q\hat{f}_y \mod \hat{L}_1$ iff

$$p\delta - q\beta = 0 = p\varepsilon_i, \ \forall i.$$

But this requires $p = 0 \Rightarrow \beta = 0$, in which case (by (6.12)) we have $\hat{f}_y = -a\hat{f}$ so that $\hat{L}_1$ is degenerate, or $\varepsilon_i = 0$, $\forall i$. As in the proof of Proposition 6.7 we therefore have $\hat{\varepsilon}_i = 0$, $\forall i$ and so $\ell \oplus \hat{L}_1 \oplus \hat{L}_2$ is constant.

∎

Observe that in $\mathbb{P}^3$ we have $x \pm y$ asymptotic co-ordinates on all four focal surfaces and so the cross-congruences are automatically W. When we are not in $\mathbb{P}^3$ it is easy to see that the cross-congruences have only one developable surface[4] and so are not congruences in terms of Eisenhart's [28] definition. In $\mathbb{P}^3$ we see that $L_1 \oplus \hat{L}_1$ has exactly two focal surfaces, for the assumption that it has any more quickly forces ℓ to be degenerate.

Summary

Since this section is quite long, and contains a lot of notation, it is worth recapping what we have proved.

- A generic isothermic line-congruence $\ell : \Sigma \to G_2(\mathbb{R}^n)$ has focal surfaces L_1, L_2. Its annihilator $\operatorname{Ann}(\ell) : \Sigma \to G_{n-2}(\mathbb{R}^n_*)$ is isothermic with respect to the same η.
- $\tilde{\ell} := \bigcap_{X \in T\Sigma} \ker \eta_X = \ell \oplus \hat{L}_0 : \Sigma \to G_{n-2}(\mathbb{R}^n)$ is isothermic with η, as is its annihilator $\ell^* := \operatorname{Ann}(\tilde{\ell}) : \Sigma \to G_2(\mathbb{R}^n_*)$.
- Given a Darboux transform $\hat{\ell} = \mathcal{D}_{-1}\ell : \Sigma \to G_{n-2}(\mathbb{R}^n)$, isothermic with closed 1-form $\hat{\eta}$, there exist unique line subbundles $\hat{L}_1, \hat{L}_2 \subset \hat{\ell}$ such that $(\hat{L}_1 \oplus \hat{L}_2, \hat{\eta})$ is isothermic.
- The annihilators $\hat{\ell}^* := \operatorname{Ann}(\hat{\ell}), \operatorname{Ann}(\hat{L}_1 \oplus \hat{L}_2)$ are also isothermic with respect to $\hat{\eta}$. Furthermore $\hat{\ell}^* = \mathcal{D}_{-1} \operatorname{Ann}(\ell)$.
- Isothermic line congruences are W-congruences, as are the dual congruences $\ell^*, \hat{\ell}^*$.
- If $\hat{L}_0 = \bigcap_{X,Y \in T\Sigma} \ker \eta_X \cap \ker \hat{\eta}_Y \subset \hat{\ell}$, then $(\ell \oplus \hat{L}_0, \hat{L}_1 \oplus \hat{L}_2)$ is *not* a Darboux pair unless $\mathbb{P}^3 := \ell \oplus \hat{L}_1 \oplus \hat{L}_2$ is constant.

6.4 Quasi-permutability in $G_k(\mathbb{R}^n)$

The discussion of the previous section was phrased entirely in terms of line congruences specifically to demonstrate that much of the structure detailed in Chapter 2 is

[4]There is only one linear combination of the derivatives $(\mathrm{f} \wedge \hat{\mathrm{f}})_x, (\mathrm{f} \wedge \hat{\mathrm{f}})_y$ which is decomposable. See Section 2.3.

preserved when one considers isothermic submanifolds of $G_2(\mathbb{R}^n)$. Much of the discussion is valid however when we consider isothermic congruences of k-planes in $\mathbb{R}^n$. Recall that the infinitesimal stabiliser of a k-plane π in the algebra $\mathfrak{sl}(n)$ is given by

$$\mathrm{stab}(\pi) = \mathrm{hom}(\mathbb{R}^n/\pi, \pi) \oplus (\mathrm{End}(\pi) \oplus \mathrm{End}(\mathbb{R}^n/\pi))_0,$$

where the first term is the nilradical. A complementary parabolic subalgebra is the stabiliser of a complementary $(n-k)$-plane to π and so $G_k(\mathbb{R}^n)$ is self-dual iff $k = n/2$.

Similarly to the previous section we will describe an isothermic bundle π of k-planes in $\mathbb{R}^n$ to be *maximal* iff $\exists$ a Darboux transform $\hat{\pi}$ such that the image of the derivative of the curved flat $(\pi, \hat{\pi})$ is a fixed maximal Abelian semisimple subalgebra of $\mathrm{stab}(\pi)^\perp \oplus \mathrm{stab}(\hat{\pi})^\perp$, necessarily of dimension $\min(k, n-k)$. In such cases the closed 1-form η will be referred to as maximally diagonal. The duality transform of the previous section generalises.

Proposition 6.10
Given a maximal isothermic submanifold $\pi : \Sigma^k \to G_k(\mathbb{R}^n)$ where $k < n/2$, then $\tilde{\pi} := \bigcap_{X \in T\Sigma} \ker \eta_X$ is a subbundle of $\Sigma \times G_{n-k}(\mathbb{R}^n)$ which is isothermic with respect to η. Conversely, given $\tilde{\pi} : \Sigma^k \to G_{n-k}(\mathbb{R}^n)$ isothermic with maximally diagonal η, we have that $\pi := \bigcup_{X \in T\Sigma} \eta_X(\mathbb{R}^n)$ is an isothermic submanifold of $G_k(\mathbb{R}^n)$, also with respect to η.

Proof This is simply a generalisation of the analysis which generated the line bundles L_i. A maximal η splits π into $\oplus_{i=1}^k L_i$ where each L_i is the image of a rank 1 element of $\mathrm{Im}\,\eta \subset \mathfrak{sl}(n)$. Thus $\pi = \bigcup_{X \in T\Sigma} \eta_X(\mathbb{R}^n)$. Similarly, any Darboux transform $\hat{\pi}$ splits as $\oplus_{i=1}^k \hat{L}_i \oplus \hat{L}_0$ where $\hat{L}_0 = \bigcap_{X \in T\Sigma} \ker \eta_X \cap \hat{\pi}$ and so $\tilde{\pi} = \pi \oplus \hat{L}_0 : \Sigma \to G_{n-k}(\mathbb{R}^n)$ is well-defined, independently of $\hat{\pi}$, as the common kernel of all the η_X. For the isothermic conditions, note that η takes values in $\mathrm{hom}(\mathbb{R}^n/\tilde{\pi}, \pi)$ which is a subspace of $\mathrm{hom}(\mathbb{R}^n/\pi, \pi)$ and $\mathrm{hom}(\mathbb{R}^n/\tilde{\pi}, \tilde{\pi})$.

■

When $k = n/2$ we clearly have $\tilde{\pi} = \pi$ and so no duality. Given a maximal isothermic congruence of $(n-k)$-planes we will write $\tilde{\pi}$ for the dual congruence of k-planes so that $\tilde{\tilde{\pi}} = \pi$. Via the correspondence $\pi \leftrightarrow \tilde{\pi}$ we can prove an interesting substitute for the Bianchi permutability of Darboux transforms in a self-dual symmetric R-space as discussed in Section 5.6.

Proposition 6.11
Let (π, η) be maximal isothermic submanifold of $G_k(\mathbb{R}^n)$ so that $\tilde{\pi} : \Sigma \to G_{n-k}(\mathbb{R}^n)$ is well-defined. Let $\pi_1 = \mathcal{D}_{s_1}\pi$, $\tilde{\pi}_2 = \mathcal{D}_{s_1}\tilde{\pi}$ be Darboux transforms such that $s_1 \neq s_2$ and $(\tilde{\pi}_2, \pi_1)$ are complementary. Let $r_u \in \mathrm{End}(\underline{\mathbb{R}}^n)$ have eigenspaces $\pi_1, \tilde{\pi}_2$ with

eigenvalues $u^{1/2}, u^{-1/2}$ respectively. Then $r_{s_2/s_1}\pi = \mathcal{D}_{s_2}\pi_1$ and $r_{s_2/s_1}\tilde{\pi} = \mathcal{D}_{s_1}\tilde{\pi}_2$. The two transforms $r_{s_2/s_1}\pi, r_{s_2/s_1}\tilde{\pi}$ are also related by the duality transform.

In the context of the remarks at the end of Section 5.6, $\tilde{\pi}_2$ is an *anti-Darboux transform* $\mathcal{D}^{s_2}\pi$ of π. Notice that we cannot suppose that $\tilde{\pi}_2$ is the dual of a Darboux transform $\pi_2 = \mathcal{D}_{s_2}\pi$, for just as in Proposition 6.8 we see that this forces $\pi_2 \cap \ker \eta$ to be constant; quotienting out reduces the problem to the self-dual $G_k(\mathbb{R}^{2k})$, and so the proposition reduces to the discussion of Bianchi permutability in Section 5.6.

Proof In the same manner as the proof of Proposition 5.10 we see that, since r_u is well-defined, the connection

$$\tilde{\mathrm{d}}^t := r_{\frac{t-s_1}{t-s_2}} \circ \mathrm{d}^t \circ r^{-1}_{\frac{t-s_1}{t-s_2}}$$

can be written $\tilde{\mathrm{d}}^t = \tilde{\mathrm{d}} + t\eta$, where $\tilde{\mathrm{d}} = r_{s_1/s_2} \circ \mathrm{d} \circ r^{-1}_{s_1/s_2}$. $\tilde{\mathrm{d}}^t$ is flat, and so $\tilde{\mathrm{d}}\eta = 0 \Rightarrow (r_{s_2/s_1}\pi, \mathrm{d}, r_{s_2/s_1}\eta)$ is isothermic. π_1 is seen to be $(\mathrm{d} + s_2 r_{s_2/s_1}\eta)$-parallel and, being complementary to $r_{s_2/s_1}\pi$, is therefore a Darboux transform $\mathcal{D}_{s_2} r_{s_2/s_1}\pi$. Similarly $(\mathrm{d} + s_1 r_{s_2/s_1}\eta)\tilde{\pi}_2$ and so $r_{s_2/s_1}\tilde{\pi} = \mathcal{D}_{s_1}\tilde{\pi}_2$. Furthermore, since $r_{s_2/s_1}\eta$ is the closed 1-form for $r_{s_2/s_1}\pi$, $r_{s_2/s_1}\tilde{\pi}$, we have that $\widetilde{r_{s_2/s_1}\pi} = r_{s_2/s_1}\tilde{\pi}$.

∎

Figure 6-1 summarises: the solid arrows are Darboux transforms, with labelled coefficients, while the inclusions represent the duality map.

$$\begin{tikzcd}
& \pi_1 \arrow[dl, leftrightarrow, "s_1"'] \arrow[dr, leftrightarrow, "s_2"] & \\
\pi \arrow[rr, dashed] & & r_{s_2/s_1}\pi \\
\cap & r_{s_2/s_1} & \cap \\
\tilde{\pi} \arrow[rr, dashed] \arrow[dr, leftrightarrow, "s_2"'] & & r_{s_2/s_1}\tilde{\pi} \\
& \tilde{\pi}_2 \arrow[ur, leftrightarrow, "s_1"'] &
\end{tikzcd}$$

Figure 6-1: Quasi-permutability of Darboux transforms in $G_k(\mathbb{R}^n)$, $k < n/2$

We will see a second example of quasi-permutability of Darboux transforms in a non-self-dual symmetric R-space in the next section.

6.5 Isothermic Submanifolds of SO(n)

Any Lie group G is a symmetric $G \times G$-space in the following manner. Let $(g_1, g_2) \in G \times G$ act on $g \in G$ via

$$(g_1, g_2) \cdot g := g_1 g g_2^{-1}.$$

The stabiliser of g is easily seen to be

$$H_g = \{(h, g^{-1}hg) : h \in G\} \cong G.$$

Thus $G \cong (G \times G)/G$ as a homogeneous space. Define an involution on $G \times G$ via $\tau_g : (g_1, g_2) \mapsto (gg_2g^{-1}, g^{-1}g_1g)$ for which H_g is clearly the fixed set. At the Lie algebra level, τ is the involution

$$\tau_g : (\xi_1, \xi_2) \mapsto (\mathrm{Ad}(g)\xi_2, -\mathrm{Ad}(g^{-1})\xi_1),$$

with ± 1-eigenspaces

$$\mathfrak{h}_g = \{(\xi, \mathrm{Ad}\, g^{-1}\xi) : \xi \in \mathfrak{g}\}, \qquad \mathfrak{m}_g = \{(\xi, -\mathrm{Ad}\, g^{-1}\xi) : \xi \in \mathfrak{g}\}.$$

Viewing $\mathfrak{h}_g, \mathfrak{m}_g$ as the fibres of bundles $\mathfrak{h}, \mathfrak{m}$ over G we see that $G \times (\mathfrak{g} \times \mathfrak{g}) = \mathfrak{h} + \mathfrak{m}$ is a symmetric bundle-decomposition (1.11).

Now specialize to $G = \mathrm{SO}(n)$. In order to make working with this example easier, we calculate the orbits of $\mathrm{SO}(n)$ on the set of isotropic n-planes in $\mathbb{R}^{n,n}$. Let $P \subset \mathrm{SO}(n, n)$ be the stabiliser of a maximal isotropic $\varphi \in \mathbb{R}^{n,n}$. It is easy to see that the nilradical of the Lie algebra of P is $\mathfrak{p}^\perp = \bigwedge^2 \varphi$ which is an Abelian subalgebra of $\mathfrak{p} = \bigwedge^2 \varphi \oplus \varphi \wedge \hat{\varphi}$ where $\hat{\varphi}$ is any complement to φ. The $\mathrm{SO}(n, n)$-orbit of φ is therefore a symmetric R-space. It is easy to see that the stabilisers of two isotropic n-planes $\varphi, \hat{\varphi}$ are complementary iff $\mathbb{R}^{n,n} = \varphi \oplus \hat{\varphi}$. Furthermore the maximal compact subgroup of $\mathrm{SO}(n, n)$ is $\mathrm{SO}(n) \times \mathrm{SO}(n)$ and so

$$\mathrm{SO}(n, n)/P = (\mathrm{SO}(n) \times \mathrm{SO}(n))/(\mathrm{SO}(n) \times \mathrm{SO}(n)) \cap P$$

as a symmetric space. We wish to see that the second quotient is in fact a copy of $\mathrm{SO}(n)$ so that $\mathrm{SO}(n) = \mathrm{SO}(n, n)/P$.

Let φ_+, φ_- be a pair of fixed definite n-planes (of signatures $(n, 0), (0, n)$ respectively): clearly $\mathbb{R}^{n,n} = \varphi_+ \oplus \varphi_-$. Any null n-plane φ has zero intersection with $\varphi_\pm$ and so is the graph of an invertible endomorphism $T : \varphi_+ \to \varphi_-$: for all $x \in \varphi_+$,

$\exists! T(x) \in \varphi_-$ such that $x + T(x) \in \varphi$. Since φ is null, we have

$$0 = (x + T(x), y + T(y)) = (x, y) + (T(x), T(y)), \ \forall x, y \in \varphi_+. \tag{6.13}$$

Consider a second null n-plane $\hat{\varphi}$: the graph of a map $\hat{T} : \varphi_+ \to \varphi_-$. Because of the choice of $\varphi_\pm$, we have a unique map $\varphi \to \hat{\varphi} : x + T(x) \mapsto x + \hat{T}(x)$. This is the restriction to φ of the map $(\mathrm{Id}, \hat{T} \circ T^{-1})$ on $\varphi_+ \times \varphi_-$. By (6.13) it is clear that $\hat{T} \circ T^{-1} \in \mathrm{O}(n) \subset \mathrm{GL}(\varphi_-)$. Thus $\mathrm{O}(n)$ acts transitively on the set of isotropic n-planes in $\mathbb{R}^{n,n}$.

$\mathrm{O}(n) \times \mathrm{O}(n)$ is the maximal compact subgroup of $\mathrm{O}(n,n)$: it is clear that $(\mathrm{Id}, \hat{T} \circ T^{-1}) \in \mathrm{O}(n,n)$ and that therefore $\mathrm{O}(n,n)$ acts transitively. Suppose now that $\varphi \oplus \hat{\varphi} = \mathbb{R}^{n,n}$ is a decomposition into maximal isotropic subspaces. Choose bases $\varphi = \langle z_1, \ldots, z_n \rangle$, $\hat{\varphi} = \langle \hat{z}_1, \ldots, \hat{z}_n \rangle$ such that $(z_i, \hat{z}_j) = \delta_{ij}$. Any orthogonal A which preserves the decomposition $\varphi \oplus \hat{\varphi}$ necessarily has determinant 1, since, with respect to the above bases, A is the same linear map on each of $\varphi, \hat{\varphi}$. $\mathrm{SO}(n) = (\mathrm{Id}, \mathrm{SO}(n))$ therefore acts transitively on the $\mathrm{SO}(n,n)$-orbit of φ and so $\mathrm{SO}(n) = \mathrm{SO}(n,n)/P$ as a symmetric R-space.

Consider the map $B : \binom{z_i}{\hat{z}_i} \mapsto \binom{\hat{z}_i}{z_i}$, which swaps $\varphi, \hat{\varphi}$. B is orthogonal, its eigenspaces being n pairs of complex conjugate lines, hence $\det B = (-1)^n$. $\varphi, \hat{\varphi}$ are therefore in the same $\mathrm{SO}(n)$-orbit iff n is even. Thus $\mathrm{SO}(n)$ is self-dual iff n is even. We can do slightly better, for suppose $\varphi, \hat{\varphi}$ are isotropic n-planes with intersection V where $\dim V = p$. $\varphi/V \oplus \hat{\varphi}/V$ is a decomposition of a $\mathbb{R}^{n-p,n-p}$ into maximal isotropic $(n-p)$-planes, and so the problem is reduced by an even number of dimensions. We have shown that there are always two $\mathrm{SO}(n)$-orbits in the space of maximal isotropic planes in $\mathbb{R}^{n,n}$. The following table shows whether various choices of maximal isotropic $\varphi, \hat{\varphi}$ are in the same or opposite orbits:

		$\dim \varphi \cap \hat{\varphi}$: even	odd
n	even	same	opp
	odd	opp	same

Consider the maximal dimension of isothermic submanifolds of $\mathrm{SO}(n)$. Suppose that $\sigma = (\varphi, \hat{\varphi})$ are a Darboux pair, and that $d\sigma$ is conjugate to a fixed semisimple Abelian subalgebra $\mathfrak{a} \subset \bigwedge^2 \varphi \oplus \bigwedge^2 \hat{\varphi}$. Since $\mathfrak{a}$ is a collection of skew-symmetric diagonalizable commuting linear operators,[5] there exists a simultaneously diagonalizing

[5]The Jordan decomposition of $T = S + N$ is the *same* for both the adjoint action of T on $\mathfrak{so}(n,n)$ and the usual action on $\mathbb{R}^{n,n}$ as in Section 4.5.

basis of null eigenvectors $v_i, v_{-i} \in \mathbb{R}^{n,n}$ where $(v_i, v_j) = 2\delta_{i,-j}$. Each v_i must be transverse to $\varphi, \hat{\varphi}$ so write $v_i = e_i + f_i$ according to the decomposition $\varphi \oplus \hat{\varphi}$ and scale so that $(e_i, f_{-i}) = 1$. It is clear that $\dim \langle e_i, e_{-i}, f_i, f_{-i} \rangle = 4$. There are at most $\lfloor n/2 \rfloor$ of these distinct 4-dim spaces. Since $\mathfrak{a}$ permutes $\langle e_i \rangle, \langle f_i \rangle$ it follows that the common non-zero eigenspaces of the adjoint action of $\mathfrak{a}$ are at most

$$\langle e_i \wedge e_{-i} + f_i \wedge f_{-i} \rangle, \quad i = 1, \ldots, \lfloor n/2 \rfloor.$$

A maximal isothermic $\varphi : \Sigma \to \mathrm{SO}(n)$ therefore has $\dim \Sigma = \lfloor n/2 \rfloor$. In such a case $\eta = \sum_{i=1}^{\lfloor n/2 \rfloor} e_i \wedge e_{-i}\omega_i$ for some non-zero 1-forms ω_i.

If n is even, then we already have a theorem of permutability of Darboux transforms since SO(n) is self-dual. If n is odd, then SO(n) is non-self-dual so none of our permutability theorems apply. Indeed, for n odd, any two Darboux transforms φ_1, φ_2 of φ necessarily have non-trivial intersection. However, similarly to Section 6.3, we can make progress by observing that the kernel of η intersects non-trivially with any Darboux transform.

Let (φ, η) be maximal isothermic with Darboux transform $(\hat{\varphi}, \hat{\eta})$ such that $\hat{\varphi} = \mathcal{D}_{-1}\varphi$ (i.e. $\mathcal{N} = \eta + \hat{\eta}$). If n is odd, the above analysis tells us that there exist line bundles $L \subset \varphi, \hat{L} \subset \hat{\varphi}$ such that

$$L = \bigcap_{X \in T\Sigma} \ker \hat{\eta}_X \cap \varphi, \qquad \hat{L} = \bigcap_{X \in T\Sigma} \ker \eta_X \cap \hat{\varphi}.$$

Indeed defining $(n-1)$-planes $\varphi_- = \bigcup_{X \in T\Sigma} \eta_X(\mathbb{R}^{n,n})$, $\hat{\varphi}_- = \bigcup_{X \in T\Sigma} \hat{\eta}_X(\mathbb{R}^{n,n})$ we have

$$\varphi = L \oplus \varphi_-, \qquad \hat{\varphi} = \hat{L} \oplus \hat{\varphi}_-,$$

where $\varphi_-^\perp = \varphi_- \oplus L \oplus \hat{L}$, $\hat{\varphi}_-^\perp = \hat{\varphi}_- \oplus L \oplus \hat{L}$, since $(\eta_X f, g) = -(f, \eta_X g) = 0$ if $g \in \ker \eta$. Since $L, \hat{L}$ is the common kernel of all $\eta_X, \hat{\eta}_X$ we have a transform of the Darboux pair $(\varphi, \hat{\varphi})$:

$$\begin{pmatrix} L \oplus \varphi_-, \eta \\ \hat{L} \oplus \hat{\varphi}_-, \hat{\eta} \end{pmatrix} \mapsto \begin{pmatrix} \hat{L} \oplus \varphi_-, \eta \\ L \oplus \hat{\varphi}_-, \hat{\eta} \end{pmatrix}.$$

While this transform gives us a new pair of isothermic congruences of n-planes, it does not, in general, give us a new Darboux pair: $L \oplus \hat{\varphi}_-$ being $(\mathrm{d} + t\eta)$-parallel (for *any* t) forces $\mathrm{d}L = 0$ (the converse similarly implies $\hat{L}$ constant) and so restricting to $(L \oplus \hat{L})^\perp$ we have that $(\varphi_-, \hat{\varphi}_-)$ are a Darboux pair in the self-dual symmetric R-space SO($n - 1$). Notice also that any orthogonal transform which fixes $\varphi_-, \hat{\varphi}_-$ and permutes $L, \hat{L}$ necessarily has determinant -1 and so the above transform permutes the dual R-

spaces M and M^*. More is true, for the map $L \oplus \varphi_- \mapsto \hat{L} \oplus \varphi_-$ is independent of the choice of Darboux transform $\hat{\varphi}$ since (mod φ_-) $L, \hat{L}$ are the unique null lines in the quotient space $\varphi_-^{\perp}/\varphi_-$. Consequently we will write $\tilde{\varphi} = \hat{L} \oplus \varphi_-$ to stress this independence. We therefore have a duality of maximal isothermic submanifolds of M with those of M^* exactly as in Proposition 6.10.

Similarly to Proposition 6.11 we may now apply the duality transform to obtain a quasi-permutability theorem for Darboux pairs in SO(n) where n is odd. Let (φ, η) be isothermic and assume that $\tilde{\varphi}$ is well-defined. Let $\varphi_1 = \mathcal{D}_{s_1}\varphi : \Sigma \to M^*$ be a Darboux transform of φ and $\tilde{\varphi}_2 = \mathcal{D}_{s_2}\tilde{\varphi}$ where we assume (generically) that $(\varphi_1, \tilde{\varphi}_2)$ are complementary and so that we can apply the dressing argument of Section 5.6. If r_u is defined in the usual way with respect to $\varphi_1 \oplus \tilde{\varphi}_2$, then

$$\hat{\mathrm{d}}^t := r_{\frac{t-s_1}{t-s_2}} \circ \mathrm{d}^t \circ r^{-1}_{\frac{t-s_1}{t-s_2}} = \hat{\mathrm{d}}^0 + t\eta$$

is flat, and so $(r_{s_2/s_1}\varphi, \mathrm{d}, r_{s_2/s_1}\eta)$ is isothermic, as is $(r_{s_2/s_1}\tilde{\varphi}, \mathrm{d}, r_{s_2/s_1}\eta)$. Furthermore

$$(\mathrm{d} + s_2 r_{s_2/s_1}\eta)\varphi_1 = (\mathrm{d}^{s_1} + (s_2 r_{s_2/s_1} - s_1)\eta)\varphi_1 = 0 \Rightarrow r_{s_2/s_1}\varphi = \mathcal{D}_{s_2}\varphi_1.$$

Since $\tilde{\varphi}_2 = \mathcal{D}_{s_2}\tilde{\varphi}$, it also follows that

$$(\mathrm{d} + s_1 r_{s_2/s_1}\eta)\tilde{\varphi}_2 = (\mathrm{d}^{s_2} + (s_1 r_{s_2/s_1} - s_2)\eta)\tilde{\varphi}_2 = 0 \Rightarrow r_{s_2/s_1}\tilde{\varphi} = \mathcal{D}_{s_1}\tilde{\varphi}_2.$$

Figure 6-2 summarises these relations.

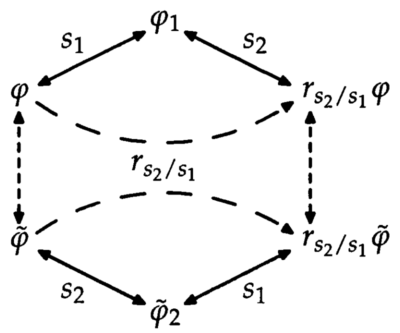

Figure 6-2: Quasi-permutability of Darboux transforms in SO(n), n odd

The solid arrows are Darboux transforms while the vertical dashed lines represent the duality transform.

6.6 Further Work

The two statements of quasi-permutability arising from the duality of maximal isothermic submanifolds of M with those of M^* when $M = G_k(\mathbb{R}^n)$ ($k \neq n/2$) or $\mathrm{SO}(n)$ (n odd) are intriguing. In both cases we get an extended flag: given a maximal (π, η) in $G_k(\mathbb{R}^n)$ where $k < n/2$ we have π contained non-trivially in $\ker\eta := \bigcap_{X \in T\Sigma} \ker \eta_X \subset \Sigma \times \mathbb{R}^n$ and so a new flag

$$\mathbb{R}^n \supset \ker \eta_x \supset \pi \left(= \bigcup_{X \in T_x\Sigma} \operatorname{Im} \eta_X \right) \supset \{0\}.$$

Similarly, given (φ, η) maximal in $\mathrm{SO}(n)$, we define $\ker\eta \subset \Sigma \times \mathbb{R}^{n,n}$ and observe that $\ker\eta \supset \varphi \supset (\ker\eta)^\perp$ thus giving the flag

$$\begin{tikzcd}[column sep=small, row sep=small]
 & \varphi \arrow[dr, phantom, "\supset"] & \\
\mathbb{R}^{n,n} \supset \ker\eta \arrow[ur, phantom, "\supset"] \arrow[dr, phantom, "\supset"] & & (\ker\eta)^\perp \supset \{0\} \\
 & \tilde{\varphi} \arrow[ur, phantom, "\supset"] &
\end{tikzcd}$$

codimension 1

The infinitesimal stabilisers of these flags are easily seen to be parabolic subalgebras $\mathfrak{p}$ of height *two* contained inside the parabolic stabilisers of π and φ respectively. Furthermore in both cases η takes values in the (Abelian) commutator of the height 2 nilradical: $\eta \in \Omega^1_\Sigma \otimes [\mathfrak{p}^\perp, \mathfrak{p}^\perp]$. In these two examples we have seen that the 1-form η is aware of more structure than just that of a symmetric R-space. Indeed η tells us that we have a bundle of height 2 parabolic subalgebras $\mathfrak{p}$

$$\underline{\mathfrak{g}} \supset [\mathfrak{p}^\perp, \mathfrak{p}^\perp]^\perp \supset \mathfrak{p} \supset \mathfrak{p}^\perp \supset [\mathfrak{p}^\perp, \mathfrak{p}^\perp] \supset \{0\}$$

for which there exist exactly two bundles of height 1 parabolic subalgebras f such that

$$[\mathfrak{p}^\perp, \mathfrak{p}^\perp]^\perp \supset f \supset \mathfrak{p}.$$

Unfortunately we have almost exhausted the examples of non-self-dual symmetric R-spaces of simple type which are easily amenable to investigation—there remains $\mathrm{Sp}(n)$ as a $\mathrm{Sp}(n,n)$-space which is similar to the $\mathrm{SO}(n)$ example, and several examples involving (real forms of) the exceptional Lie groups E_6, E_7—and so it is difficult to construct further examples to see if this is a general phenomenon. One of the difficulties in trying to analyse these examples is that the height 2 parabolic subalgebras $\mathfrak{p}$ defined above do not appear to depend on the kernel of η as a subset of the *algebra* $\mathfrak{g}$

in any particularly informative way. Indeed in both examples we have

$$\mathfrak{p} \supset \left\{ \begin{array}{c} \ker \eta \\ \not\pitchfork \\ (\ker \eta)^{\perp} \end{array} \right\} \supset \mathfrak{p}^{\perp}.$$

We may still ask the following questions:

- Given (f, η) maximal isothermic in a non-self-dual M, does η always see extra structure in the form of a bundle of height 2 parabolic subalgebras $\mathfrak{p}$ such that $[\mathfrak{p}^{\perp}, \mathfrak{p}^{\perp}]^{\perp} \supset f \supset \mathfrak{p}$?
- If so, is there a second bundle $\tilde{f}$ of height 1 parabolic subalgebras constrained such that $[\mathfrak{p}^{\perp}, \mathfrak{p}^{\perp}]^{\perp} \supset \tilde{f} \supset \mathfrak{p}$?
- Is $\tilde{f}$ unique?
- Given that $\mathfrak{p}$ is a map into an R-space of height 2 for which there exists a closed 1-form η taking values in the Abelian part of the nilradical of $\mathfrak{p}$, is there a sensible general theory of 'k-isothermic' submanifolds: immersions $f : \Sigma \to M$ into an R-space of height k such that there exists a closed 1-form in the Abelian part of the nilradical of f? We can at least partially answer this, for the construction of T-transforms certainly works. The obvious definition of Darboux transform $((\mathrm{d} + t\eta)\mathfrak{q} = 0$ for $\mathfrak{q} \in \Gamma(\Sigma \times M^*))$ does not however satisfy the property that $(\mathfrak{p}, \mathfrak{q})$ is a curved flat in the space of complementary pairs.

Since this thesis began and almost ended with discussions of line congruences and we have repeatedly mentioned links with congruences of spheres, it is natural to consider the following:

- Investigate isothermic sphere congruences in S^3 (or hyperspheres in S^n). Lie sphere geometry provides a bijection of the set of such congruences with isothermic submanifolds of the quadric $\mathbb{P}(\mathcal{L}^{n+1,2})$. We have already discussed isothermic maps into quadrics, but have not considered the geometric properties of such maps viewed as congruences in S^n. For example we have seen that isothermic line congruences have several interesting properties: what are the analogues in Lie sphere geometry? An interesting example of isothermic sphere congruences which are easy to construct are the isothermic *surfaces* in $S^n \cong \mathbb{P}(\mathcal{L}^{n+1,1})$ viewed as congruences of 0-spheres: since $\mathbb{R}^{n+1,1} \subset \mathbb{R}^{n+1,2}$ we have $\ell \wedge \ell^{\perp}_{\mathbb{R}^{n+1,1}} \subset \ell \wedge \ell^{\perp}_{\mathbb{R}^{n+1,2}}$ and so a 0-congruence is isothermic in $\mathbb{P}(\mathcal{L}^{n+1,2})$ with respect to the *same* η.

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